Who to share information with? Persuasion in social networks*

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Abstract

We study the information targeting and design problem of a sender in a setting where multiple receivers can communicate with each other through links in a social network. The sender commits to an information structure and chooses one of the receivers to initially observe the realised signal. The receivers simultaneously choose an action based on the information they possess, but are subject to peer effects. Each receiver wants her neighbours to choose the same action as her, and can strategically communicate her information to them. This triggers a strategic diffusion process through the network, with the realised signal reaching only a subset of receivers. The sender's objective is to persuade as many receivers as possible to take a high action level. We characterise the optimal information and targeting strategy of the sender, highlighting the fundamental trade-off between information precision and extent of diffusion. Applications include political campaigns, lobbying, and raising participation in experimental settings . Contrary to common intuition, targeting the "most central" agent in the network may not be optimal. The notion of centrality, which we term "influence" itself is endogenously determined by communication strategies and the information quality. We also conduct some comparative static analysis to understand how notions like homophily and network density affect the optimal policy of the sender.

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1 Introduction

The dissemination of information plays a critical role in society, aiding people to make a plethora of decisions ranging from buying a new product to choosing a new government. The role of social links, and societal structure in general, in this process has long been recognised by politicians and businessmen alike. Success of political activities such as lobbying and whisper campaigns invariably depend on the politicians' ability to communicate with each other and spread their agenda. Firms have used word-of-mouth communication to advertise their products to potential customers as well as to test out "β"-versions of a new product.[1] Social linkages play also play a crucial role in designing field experiments. Researchers spend considerable time raising awareness of the program to induce participation among people. Large scale experiments like microcredit programs in rural areas invariably depend on social connections among villagers to spread this awareness, making it an important component of the overall experiment design (see for example Banerjee, Chandrasekhar, Duflo, and Jackson (2013)). Analysing the economic motives underlying such information spread can thus guide researchers to improve large-scale experiment design.

There are two important aspects underlying the dissemination process of the above examples that merit discussion: (a) what kind of information is send, and (b) who this information is targeted to. Each of these issues have been studied extensively in isolation. The "what kind of information to send" aspect has been a subject of extensive focus in the economics literature, especially following Kamenica and Gentzkow (2011) and the subsequent development of the information design field. The targeting aspect has been studied in the computer science literature to analyse how malicious codes diffuse through servers or in epidemiology to study how viruses are transmitted. A complete understanding of information dissemination through society, however, needs to analyse both these aspects in conjunction. The kind of information to be shared shapes how it is targeted. Suppose the Nature Conservancy group aims to lobby Senators to introduce a new carbon tax. Since the agenda corroborates with the Democratic

[1] An article published in Nature talks about how production houses have used word-of-mouth to enhance viewership of their films. This channel became a critical source for many production houses during the pandemic period of 2020-21 where most theatres across the world remained shut for more than a year. Recently, there have been more concerted effort on the behalf of firms to understand the underlying network structure in society. Referral coupons ("*get a discount if 5 of your friends sign up*") is a common example of such an effort.

agenda on climate change, relatively little effort is required to get the Democrat senators on board. However extensive reports and costs are required to convince the Republicans. Thus the accuracy of information clearly plays a role in how it is targeted. Conversely, poor targeting may render even accurate information useless. A month before the Indian general elections in 2019, the Indian National Congress party (INC) announced the NYAY (universal basic income) scheme that guarantees a lump-sum amount of INR 72,000 per annum to poor households. With the country facing economic stress, this was a welcome initiative and the INC developed extensive advertisements for the project. The INC, to counter the incumbent BJP juggernaut, ran an expensive but poorly targeted awareness campaign - it turned out that the largest beneficiaries about the project (the poorest households) were among the least aware of the project. Dismal targeting ultimately contributed to the Congress party's crushing defeat at the ballots a month later. It must be noted that the underlying assumption in each of these examples is the fact that it costly to communicate with each and every consumer. These costs may be due to legal obligations, time and effort costs, and even include psychological or attention costs.

The experience of the INC points to a third important aspect that distinguishes information diffusion from other kinds of diffusion typically studied in the literature. Unlike viral diffusion, where the transmission occurs mechanically through contact, strategic motives play an essential role in information sharing. This chapter studies the information dissemination problem by combining the three inter-related aspects through the lens of a game theoretic analysis. In this chapter I study a stylised model of private bayesian persuasion between a sender and multiple receivers connected in a social network. There is an unknown state of the world and two kinds of actions that the receivers can take - high and low. The sender wants to persuade the receivers to take the high action. The receivers have heterogeneous preferences and the payoffs depend on the actions of their friends. The sender can privately communicate some information about the state to a strict subset of receivers. The latter then decides whether or not to transmit this piece of information without falsifying it to his friends in the social network. Once communication ceases, each receiver takes an action to maximise his payoff given whatever information he receives.

The analysis of the model involves two main parts. I first characterise the sender's choice of information structure in terms of the network and the bias of the receivers.

Secondly I show how the optimal choice varies as I vary the parameters of the model. While inter-related, the bias and the network structure enter the sender's payoff through distinct channels. The bias affects the incentives to communicate and hence impacts the precision of information sent by the sender. On the other hand, the network determines how many receivers choose the high action. A precise information structure, by resolving uncertainty, reduces the sender's payoff but it increases payoff through by maximising spread of the information through the network. This trade-off characterises the targeting strategy followed by the sender through a "centrality" measure that is endogenously determined in equilibrium. We term this as a receiver's "influence". It turns out that this measure is related to the concept of diffusion centrality defined in Banerjee et al. (2013) adapted to allow for endogenous communication among the nodes. When all the receivers in the model have the same bias, "influence" of a receiver reduces to the diffusion centrality of that node. This model nests a number of existing models in the information design literature. An empty network represents a single receiver persuasion model of Kamenica and Gentzkow (2011). On the other hand, a complete network with homogeneous receivers nests the case of public persuasion (theorem 4 of Arieli and Babichenko (2016)). Finally, I study the comparative statics of the sender's choice by varying the magnitude of bias, network structure, and the proportion of biased receivers. As the bias, and proportion of biased receivers increase, it becomes increasingly difficult to spread information. At some point, the sender targets only the biased receivers and ensures they take the high action. Lastly, I show that the the sender's payoff is monotone with respect to an appropriate order on the network structure with homogeneous receivers. The complete network is always the best from the sender's perspective.

1.1 Related literature

This paper contributes to two distinct strands of literature. It is related to models of strategic information transmission in networks (Galeotti, Ghiglino, and Squintani (2013), Bloch, Demange, and Kranton (2018), Hagenbach and Koessler (2010), Chatterjee and Dutta (2016) etc). Galeotti et al. (2013) study a model of cheap talk among players embedded in a network. They characterises the condition for truthful communication networks that emerge in a beta-binomial setup and exogenous signal precision. Bloch et al. (2018) study the transmission of (possibly false) information in a network. A randomly selected receiver (whose identity is unknown) learns the true state and spreads a rumour about the state. This rumour can then transmit through the network. They provide an algorithm the maximal truthful communication network. My paper, on the other hand, studies how an outsider can influence the truthful networks that emerge in equilibrium by controlling information precision and who receives information initially. Chatterjee and Dutta (2016) study a model of product diffusion where a firm can seed the network with "innovators" who influence the receivers to buy the product (of unknown quality). I study a similar diffusion problem, but instead allow the sender to seed the network with some information about the quality. Moreover, unlike Chatterjee and Dutta (2016), diffusion in my model involves a strategic interaction among and receiver and his neighbour. A very closely closely related paper is Galperti and Perego (2019).[2] They study a model where an outsider can manipulate beliefs of receivers playing a game in a network by designing information. They characterise the distribution of outcomes that the sender can generate, but don't discuss the targeting problem of the sender. My paper, at the cost of additional structure, complement their work by focussing on the targeting problem. Moreover the outsider in the two papers face diametrically opposite forces: the outsider in their model benefits by keeping information local, while the outsider in my paper benefits from more spread.

On the other hand, this paper adds to the literature on bayesian persuasion with multiple receivers (Wang (2013), Alonso and Câmara (2016), Arieli and Babichenko (2016) etc). These papers assume that the sender can send private information to all receivers, and the receivers do not communicate their private information. This paper relaxes these assumptions to study how the sender incorporates the ability of receivers to communicate in the optimal information structure. Lastly, my paper contributes to the burgeoning literature on the market for news. There is a large body of empirical literature measuring the bias present in news reports (Gentzkow and Shapiro (2010), Chiang and Knight (2011)) and how they impact voting outcomes (DellaVigna and Kaplan (2007), Puglisi (2011)). On the other hand, Gentzkow and Shapiro (2006) provides a simple theoretical model to explain how media houses distort news to cater to consumer political tastes. This paper adds to the literature by studying the role that bias plays in the diffusion of news.

[2] Kranton and McAdams (2019) presents a similar model to this paper but does not deal with the targeting problem. This is unlike my paper where targeting forms the core of the problem.

2 Stylised example

The following simple example serves to illustrate the key insights of the model. A researcher wants to persuade Ann and Bob to take part in an experiment. The outcome of the experiment may be good (G) or bad (B), and everyone believes that the outcome is good with probability 0.2 (i.e. $Pr(G) = 0.2$). The researcher, however, can try to persuade them to participate by sending signals $S = \{g, b\}$ that provides some information about the outcome. This signal, however, can be sent to either Ann or Bob, but not both. Thus the researcher has two decisions to make: (a) accuracy of the signals, and (b) a choice of who receives the information. Accuracy of the signal structure is controlled by a choice of the conditional distribution over S, which can be summarised in the table below:

		S: b	g
ω	G	$\pi(b\|G)$	$\pi(g\|G)$
	B	$\pi(b\|B)$	$\pi(b\|B)$

Table 1: Information structure describing the accuracy of the signals

The researcher's objective is to maximise participation into the experiment. Based on the information they receive (or not receive), Ann and Bob simultaneously decide whether to participate (p) or not participate (np). Both Ann and Bob wants to participate if and only if the if the outcome is good. Their payoffs are given in the matrix below

		a: np	p
ω	G	0	1
	B	0	−1

(a) Ann's payoff: $u(a, \omega)$

		a: np	p
ω	G	0	2
	B	0	−1

(b) Bob's payoff: $v(a, \omega)$

Table 2: Payoff matrix of the players for each state-action pair

Ann and Bob's optimal actions can be characterised easily. Let $\mu(s)$ be the posterior probability of the outcome being good following a signal s, i.e. $\mu(s) = Pr(G|s)$. If someone does not observe any signal, then they do not update their priors, i.e. $\mu(\emptyset) = 0.2$. Thus following the signal s,

Ann : participates if and only if $\mu(s) \geq \frac{1}{2}$

Bob : participates if and only if $\mu(s) \geq \frac{1}{3}$

It is straightforward to see that any optimal information structure has the property that $\pi(g|G) = 1$. This holds irrespective of who the outlet chooses. Each signal in equilibrium conveys an action recommendation to the readers: for example, a signal g recommends Ann or Bob to participate in the program. Thus if the outcome of the experiment is actually good, the researcher has no incentive to send a bad signal b. This implies that $\pi(g|G) = 1$, and thus $\mu(b) = 0$. The optimal strategy of each reader following a r signal is not to click. This simplifies the researcher's problem to simply choosing $\pi(g|B)$ (or equivalently $\mu(g)$).

Finally we assume that both Ann and Bob are connected in a social network.

Empty network Suppose Ann and Bob have no connections between them. Irrespective of who the researcher targets, she can get at-most 1 participation in this case. Note

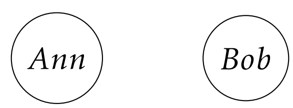

FIGURE 1: An empty network

that only needs to induce a posterior $\mu(g)$ of 0.33 (and thus $\pi(g|B) = \frac{1}{2}$) to convince Bob to participate. The expected payoff from choosing Bob is thus $Pr[g]1 + Pr[b]0 = (1*0.2+0.5*0.8) = 0.6$. The corresponding payoff from choosing Ann yields an expected payoff of 0.4 to the researcher. Hence Bob is clearly a better choice in this case.

Partial network Consider the case where Ann is connected to Bob, and not vice versa. This represents a situation where the person who is easier to persuade is less connected. Following the analysis of the previous case, the researcher gets 0.6 from choosing him.

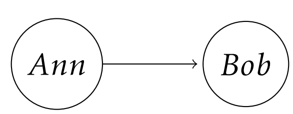

FIGURE 2: A line network

This is because Bob has no connections to Ann. Now consider the case where the researcher chooses Ann. Notice that Ann participates when the posterior $\mu(g) \geq 0.5$. Owing to the link from Ann to Bob, the latter receives the signal that Ann observes. In this range for $\mu(g)$ Bob finds it optimal participate as well. Hence the researcher gets two participation whenever $\mu(g) \in [0.5, 1]$, and 0 otherwise. It is easy to show that targeting Ann yields a payoff value of $[0.2 \times 1 + 0.8 \times 0.25]2 = 0.8$ to the outlet. Thus targeting Ann is optimal in the partial network setting, despite Bob being the easier to persuade.

Complete network Now let us consider the other extreme - both Ann and Bob are connected to one another. The analysis when the researcher chooses Ann is exactly

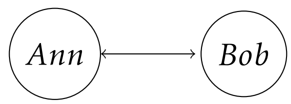

Figure 3: A complete network

as in the partial network case, yielding an expected payoff of 0.8. Now suppose the researcher chooses Bob. In this case, Ann observes whatever signal that Bob observes. For any $\mu(g) \in [0, 0.33)$, neither Bob nor Ann participates in the experiment. Hence the researcher's payoff is 0. When $\mu(g) \in [0.33, 0.5)$, Bob participates but Ann does not. The researcher gets only one participation in this case. For $\mu(g) \in [0.5, 1]$ on the other hand, both Bob and Ann are willing to participate. There is full participation in this case. It is easy to show that it is optimal for the researcher to choose $\mu(g) = 0.5$ when Bob is targeted. This yields the outlet an expected payoff of 0.8, leaving the her indifferent between targeting Ann and Bob.

Remarks Despite its stylised nature, the example illustrates the key trade-off between information accuracy and optimal target choice. When communication is constrained by the network, as in the partial network case, the researcher optimally choose a more accurate information structure (which reduces ex-ante payoff) to maximise spread of information (which weakly increases payoff). Moreover notice that in the complete network case, the optimal signal accuracy was more than is required to persuade the target. It is as if the researcher is indirectly targeting Ann, with Bob playing the role of an information intermediary. Lastly there seems to be a natural ordering of networks from the outlet's perspective: he prefers the complete (or partial) network over the empty network. This is in stark contrast to Galperti and Perego (2019) where the designer strictly prefers the empty network over any other network. The general model developed in this paper formalises this conjecture for an appropriate ordering over the network structures (theorem 2).

This example ignores the ability of Ann or Bob to hide information from one another. The ability to communicate and hide information is relevant in many settings of interest, like political campaigns. People are often hesitant to spread information that is detrimental to their preferred political party. This shapes how far information can spread in a network, consequently affecting the information design and targeting problem for the researcher. We allow for the possibility of communication in the general model developed below and characterise how these strategic incentives shape the optimal target choice.

Layout The rest of the paper proceeds as follows. I highlight the key forces of the model in terms of a simple example in Section 2. Section 3 describes the model and characterises the solution under general specifications, while section 4 analyse the sender's optimal targeting problem and perform welfare analysis. I provide some comparative static results in Section 5. Section 6 concludes and mentions some direction of future research.

3 The model

Environment The model consists of a finite set of receivers $N = \{1, 2, \ldots, n\}$ with $n \geq 2$, and a sender. The receivers are embedded in a directed network whose links are defined in the set $G \subseteq N^2$. A link $ij \in G$, often represented by $i \to j$, indicates that receiver j can listen to receiver i, but not vice-versa. The network thus describes the exogenous constraints to communication. Given G, the out-neighbourhood G_i^{out} is the set of receivers with can listen to receiver i.[3] Similarly, the in-neighbourhood G_i^{in} is the set of receivers who i listens to. I define G_i to be the set of receivers who can (potentially) receive any information that receiver i possesses, that is $G_i = \{j \in N : \exists \text{a path from } i \text{ to } j\}$. The sender, on the other hand, is a third party outside the network but can perfectly observe it. The unknown state of the world is binary $\Omega := \{0, 1\}$ with a common prior belief $\mu_0 = Pr(\omega = 1) \in (0, 1)$.

The game proceeds as follows. The sender first commits to disclose some information

[3] Readers unfamiliar with the graph theoretic terms may refer to Chapter **??** for a basic introduction and the references therein.

about the state to a set of "seeds" in the network.[4] The information is privately observed by the seeds, who can then communicate this privately to their neighbours. Discrete rounds of communication diffuses the seeded information through the network. Following this phase, all the receivers simultaneously choose an action based on the information they have received.

Information design stage The sender commits to reveal some information about the state and aims to diffuse it through the network. He can influence this diffusion by controlling (1) the correlation between the signals and states, and (2) through the choice of the seeds. He is only constrained in the number of seeds he can choose.

Suppose the sender can choose one of the receivers $t \in N$ as the "seed" for the information diffusion process. In other words, receiver t privately observes the signal realised from the designer's chosen information structure at the beginning of the game. His choice can then be formalised in terms of an information structure $\{t, (S, \pi)\}$ where S is a finite set of signals and $\pi : \Omega \to \Delta(S)$ is the conditional probability distribution over the signal profiles. For convenience we often denote $\{t, (S, \pi)\}$ as (S_t, π) throughout the paper, and let χ be the set of all information structures. The sender commits to an information structure (S_t, π) before the state ω is realised. All receivers observe the chosen information structure, but only the seeds initially observe their respective signal realisation.

Communication stage Information transmission in the network can be described in the following way. After the sender chooses his strategy, there is a finite number of communication rounds. In each round, receiver i sends a message to each of his neighbours j from a finite set M_{ij}^0.[5] I assume that there are at least $n-1$ communication rounds and $S_t \subseteq M_{ij}^0$ so that the only physical constraint to communication is the network structure.

A communication strategy for each receiver is a mapping from the set of histories he observes to the set of possible messages. The initial history observed by each receiver i is of the form $h_i^0 = \{s_i\}$ if $i = t$ and $h_i^0 = \emptyset$ otherwise. The message sent by receiver i

[4] A seed is a receiver in the network to whom the sender provides information to trigger a diffusion process.

[5] The message space depends on the sender's choice of information structure, i.e. on (S_t, π). I omit this dependency for the brevity of notation.

to j following history h_i^0 is $M_{ij}(h_i^0) \subseteq M_{ij}^0$. The history observed by i in round τ can be recursively written as $H_i^\tau = \left\{ \{h_i^{\tau-1}, m_i\} : h_i^{\tau-1} \in H_i^{\tau-1} \text{ and } m_i \in \times_{j \in G_i^{in}} M_{ji}(h_j^{\tau-1}) \right\}$. The set of all possible histories for i is $\mathcal{H}_i = \cup_{\tau=1}^{n-1} H_i^\tau$. A communication strategy can be described by a mapping $\sigma_i : \mathcal{H}_i \to \Delta(\times_{j \in G_i^{out}} M_{ij}^0)$. In this chapter I assume that all information is tagged with the source which cannot be falsified: receiver i may lie about receiver k's information content to j, but cannot claim that the information was sent by some other receiver $l \neq k$.[6] Moreover receivers are restricted to communicate in round τ only if he received some new information in round $\tau - 1$. This implies that a receiver who decides to block a message to his neighbour cannot masquerade as if he did not possess any information about that message. This is formalised as follows. Let $\top_\tau = \{i \in N : h_i^{\tau-1} \neq h_i^{\tau-2}\}$ be the set of receivers who receive some new information in round τ. They are the set of active receivers in the network in round τ. $N \backslash \top_\tau$ is the set of passive receivers.[7]

Assumption 1. *receiver i's communication strategy σ is defined only in communication rounds τ, in which he is active.*

Additional structure on the set of available messages $M_{ij}(S_t, \pi)$ depends on the particular communication protocol we are interested in. In particular, I'll assume that receivers can engage in cheap talk, and delegate the case of verifiable transmission to the Appendix.

Definition 1. *Under cheap talk, the set of possible messages that each receiver can transmit upon receiving signal(s) s is unrestricted:*

$$M_{ij}(h_i^\tau) = \times_{k \in G_i^{in}} M_{ij}$$

Under verifiable transmission, receiver i can either truthfully transmit the k's message or block it to receiver j in round τ:

$$M_{ij}(h_i^\tau) = \times_{k \in G_i^{in}} \{m_{ki}, \emptyset\} \text{ where } (m_{ki})_{k \in G_i^{in}} \in h_i^{\tau-1} \backslash h_i^{\tau-2}$$

[6]If i wants to convey k's private information s_k to receiver j, the message in our model would be as follows: m_{ij}: k has sent signal s "

[7]One can easily dispense with the assumption, but impose an intuitive criterion to rule out equilibria where such deviations are possible in equilibria. See Foerster (2019) for an application of such refinements in network settings.

The following example illustrates the communication protocol.

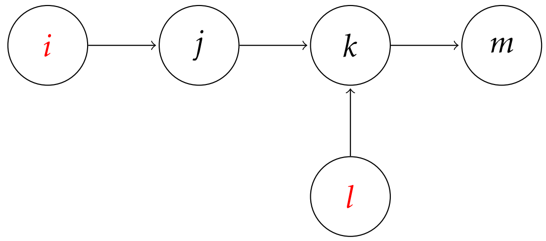

FIGURE 4: A situation when receivers i and l possess signals s_i and s_l respectively.

Example 1. Consider the graph in figure 4 and suppose the sender seeds receivers i and l with information. In round 1, only receivers i and l are active (with $h_i^0 = (s_i)$ and $h_l^0 = (s_l)$) while the others are passive (who observe empty histories $h_j^0 = h_k^0 = h_m^0 = \emptyset$). receiver i sends $m_{ij} \in M_{ij}(h_i^0)$ to j and l sends $m_{lk} \in M_{lk}(h_l^0)$ to k in this round. Thus at the end of round 1, j and k observe histories $h_j^1 = (m_{ij})$ and $h_k^1 = (m_{lk})$ respectively.

The active receivers in round 2 are thus j and k, while the passive receivers are i, l and m. receiver j sends a message $m_{jk} \in M(h_j^1 \backslash h_j^0) \equiv M(m_{ij})$ to receiver k sends a message $m_{km} \in M(h_k^1 \backslash h_k^0) \equiv M(m_{lk})$. The observed histories are as follows: $h_i^2 = h_i^1$, $h_l^2 = h_l^1$, $h_j^2 = h_j^1$, $h_k^2 = (m_{lk}, m_{jk})$, and $h_m^2 = (m_{km})$.

In round 3, receivers k and m are active. Notice that m has no neighbours to communicate to, and hence has no communication strategy. As for receiver k, the new information he received in round 2 is $h_k^2 \backslash h_k^1 = (m_{jk})$. Thus he sends a message $m_{km} \in M(m_{jk})$ to receiver m. This exhausts the communication possibilities in this simple example.

Action stage At the end of the communication stage, each receiver observes the history h_i^{n-1} and chooses an action a_i from a compact metric space A. This yields a payoff of

$$u_i(\mathbf{a}, \omega) = -(a_i - \omega - b_i)^2 - \underbrace{\sum_{j \in G_i^{out}} (a_j - \omega - b_i)^2}_{\text{Network Externality}} \tag{1}$$

Two aspects of the the receiver's preference merits discussion - the bias parameter b_i, and the network externality part. receiver i has single peaked preferences with respect to his own actions - he wants to choose an action as close as possible to his bliss point

$\omega + b_i$. The bias b_i can take one of two values: $b_i \in \{0, b\}$ with $b > 0$. The set of "biased" receivers is $B \equiv \{j \in N : b_j = b\}$ whereas $U = N \backslash B$ is the set of "unbiased" receivers. It is straightforward to see that a biased receiver's optimal action is always higher than that of an unbiased receiver following any history. In this sense, the biased receiver's preference is relatively more aligned towards the sender's preference. The network externality component of the payoff implies that receiver i wants his neighbours to choose an action as close as possible to his own bliss point.Thus the action strategy of receiver i is a mapping $a_i : H_i^{n-1} \to A$. I assume that for any a_{-i}, there exists a unique maximiser $\mathbf{a_i^*}$ of $u_i(a_i, \mathbf{a_{-i}}, \omega)$.[8] The fact that receiver i's payoff depends only on the payoff of his neighbours (G_i^{out}) vastly simplifies receiver i's optimal strategy. In particular, receivers do not require the knowledge of the entire network when making their action and communication strategies.

The sender has state independent preferences and cares only about the actions taken by the receivers. His payoff function is denoted by a function $v : A \times \Omega \to \mathbb{R}$ which is increasing in a_i for all i. The sender only cares whether each receiver takes a high enough action, irrespective of the state. Let $A_\kappa = \{j \in N : a_j \geq \kappa\}$ be the set of receivers who choose an action of at least a threshold κ.[9] His payoff is given by $v(A_\kappa) = |A_\kappa|$. Without loss of generality, I normalise the payoff $v(\emptyset) = 0$. Thus there are effectively two actions from the sender's perspective.

Equilibrium It is well known that bayesian updating in social networks is notoriously complicated. Inferring others' private information in the network may require him to keep track of very complicated strategies and paths through which information can flow (see Levy and Razin (2018), Mueller-Frank (2013)). To avoid such complications, I restrict the inferences that receivers who are never communicated with in the network can make about the state. Let σ denote the profile of communication strategies.

Assumption 2. *Fix a seed t and consider a receiver $j \notin \cup_{i \in T} G_i$. The posterior formed by receiver j following the communication phase is equal to the prior μ_0 independent of σ.*

Assumption 2 has no bite if the underlying network is strongly connected. Consider figure 5 where receiver i is the only seed and hence $k \notin G_i$. A fully sophisticated receiver k may be able to infer something about the state by predicting the communication

[8]Separability in $\mathbf{a}$ implies that a_i^* is also the maximiser of the function $f_i(a_i, \omega)$

[9]I refer to an action $a_j \geq \kappa$ as a "High" action.

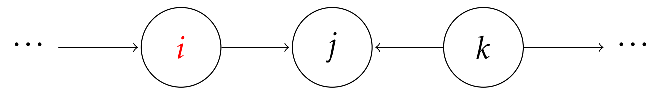

FIGURE 5: A path along which signal s_i is blocked along the path.

strategies and choice of information structures. This would induce k to update his prior despite being a *"passive observer"* in the communication phase. Assumption 2 prevents the receivers from performing any such inference.

An analyse the weak perfect bayesian equilibrium of the above game, in conjunction with the belief restrictions just described.

Definition 2. *An equilibrium consists of the tuple* $\left((S_t^*, \pi^*), \{\sigma_i^*, a_i^*\}_{i\in N}\right)$ *such that:*

1. *Given* $\{\sigma_i^*, a_i^*\}_{i\in N}$, $(S_t^*, \pi^*) \in argmax_{\pi\in\chi}\ \mathbb{E}_\pi[v(\mathbf{a}^*, \omega)]$
2. *Given* (S_t^*, π^*) *and* $\{a_i^*\}_{i\in N}$, $\sigma_{ij}^* \in\ argmax_{\sigma_{ij}\in M_{ij}} \mathbb{E}_{\pi^*,\sigma_{-\mathbf{i}}^*}[u_i(\mathbf{a}^*, \omega)]$*; posteriors are formed in accordance to Bayes' rule wherever possible .*
3. *Given* (S_t^*, π^*) *and* $\{\sigma_i^*\}_{i\in N}$, $a_i^* \in\ argmax_{a_i\in A}\ \mathbb{E}_{\pi^*,\sigma^*}[u_i(a_i, \mathbf{a}_{-\mathbf{i}}^*, \omega)|h_i^{n-1}]$.

I maintain the standard tie-breaking assumption where each receiver chooses a strategy in favour of the sender whenever indifferent.[10]

3.1 Discussion of the model

Before analysing the optimal strategies of the players, I briefly discuss the key aspects of the model and how they impact the analysis. The first important point to note is that the receivers play a "trivial" game in the action stage. That is, the receiver's optimal action is independent of the action choice of the other receivers. The only strategic aspect among the receivers occurs during the communication phase. Note that the seed (and any other receiver possessing information) act as information intermediaries in my environment. The information the intermediary holds provides action recommendation to him as well as those to whom he transmits this information. Consequently he can predict his neighbour's equilibrium actions perfectly, and can thus contingent possible deviations on this added information. The trivial game ensures that this does not impact the intermediary's optimal actions. This logic is no longer true when the receivers are

[10] In particular it implies that receivers always transmits signals whenever they are indifferent between transmitting and not transmitting, and always choose the "high" action whenever the posterior makes him indifferent between the two types of actions.

playing a strategic game in the action stage. This has important consequences for the sender's incentives. Specifically, the same signal profile can be used to incentivise receivers to take high actions. The sender thus aims to maximise the spread of signals in the network, which is opposite to the forces present in Galperti and Perego (2019).

The second major assumption is the fact that the sender commits to an information structure before the state is realised. While this is a standard assumption in the information design literature, it is worthwhile discussing it in the context of this paper. Since the sender has state independent preferences, he has an incentive to reveal only favourable signals to the seed. receivers would take this into account and ignore the sender's information. Hence the commitment assumption conveys meaning to signals in equilibrium. Somewhat more difficult task is to find such commitment devices employed in practice. One commonly used justification stems from long-run reputation concerns for media houses. For example, people are less likely to believe on well established fake news site (e.g. InfoWars) compared to a news outlet like Fox News or New York Times. This gives incentive for media houses to commit to reveal unfavourable news to the public. Related to this is recent attempts of digital platforms to crackdown on the spread of fake news (Allcott and Gentzkow (2017), Allcott, Gentzkow, and Yu (2019)).[11] These attempts have intensified following the 2016 US elections and the Cambridge Analytica scandal. There are websites that maintain lists of websites known to spread fake news.[12]

A lot of the analysis and results later on in this paper uses additional structure on the model. This includes basically using some well known functional forms for the receivers' utilities and structure on the possible messages that receivers can send during the communication phase. The aim of these additional structures is essentially to obtain a tractable characterisation of the communication strategies and posterior beliefs of the receivers. At the end of the communication phase, receivers either know the information perfectly, or knows nothing at all. Similarly, the fact that a receiver's payoff depends only oh his neighbour's actions simplifies the communication strategy - receivers don't require knowledge of the the entire network structure in computing his optimal strategies.

[11] http://fortune.com/2016/12/15/facebook-fake-news-crack-down/.

[12] https://www.politifact.com/punditfact/article/2017/apr/20/politifacts-guide-fake-news-websites-and-what-they/

4 Equilibrium analysis

I analyse the targeting problem in this section by imposing additional structure on the model. Separability of **a** implies that the optimal action of each receiver is independent of others' actions. Consequently the receivers play a "trivial game" in the Action Stage. The only strategic component occur during the communication phase. Note that receiver i can convince his neighbour j to shift his action towards i's bliss point only through information transmission (and not through action choice). A communicating receiver thus plays a sender-receiver game with each of his neighbours: the holder of information (or information intermediary) is the sender, while his neighbours are the receivers. Moreover the following parametric assumption is maintained throughout this section

Assumption 3. *The magnitude of the bias b belongs to the interval $\left(\frac{\kappa}{2}, \kappa - \mu_0(\omega = 1)\right)$*

This ensures that all the receivers choose actions $a_i < \kappa$ under the prior. Under assumption 3, we can characterise the optimal actions taken by the receivers for any arbitrary information structure. This is formalised in lemma 1 below.

Lemma 1. *Fix an information structure (S_t, π). The optimal action of receiver i following a non-null history h_i^{n-1} where he observes signal s is given by*

$$a_i^* = \mu^i(\omega = 1|s) + b_i \tag{2}$$

where $\mu^i(\omega = 1|s) = \frac{\prod_{k=0}^{K-1}\left[\sum_{s' \in S_t} \sigma_{i_k i_{k+1}}(s|s')\pi(s'|1)\right]\mu_0(\omega=1)}{\sum_\omega\left\{\prod_{k=0}^{K-1}\left[\sum_{s'} \sigma_{i_k i_{k+1}}(s|s')\pi(s'|\omega)\right]\right\}\mu_0(\omega)}$

Given the communication protocol, a receiver either has no information ($h_i^{n-1} = \emptyset$) or all the information flowing through the network ($h_i^{n-1} = (S_t)$). This simplifies the posterior belief calculation for the receivers. I solve the model under the cheap talk protocol, and delegate the results from the hard information protocol to the appendix.

This protocol allows a receiver to costlessly falsify the content of messages he has received through communication. As discussed in section 3, he cannot falsify the source of information. For example, if receiver i (truthfully) learns about k's signal s_k, he can costlessly tell his neighbour that k's information was $\hat{s}_k \neq s_k$ (as long as $\hat{s}_k \in$ **supp** π). But i cannot deviate to a message s_l where $l \neq k$. The set of possible messages that an active receiver i can send to his neighbour j in any round τ is given by $M_{ij}(h_i^\tau) = \{S_t \in S_t : S_t \in \textbf{supp } \pi\} \equiv \bar{S}_t$. Thus the communication game between receiver i and each of his neighbour j reduces to a simple variation of Crawford and Sobel (1982) with a finite set of "states". I restrict communication strategies to two simple forms: truthfully revealing all signals or complete babbling. That is, $\sigma_{ij}(s) = s$ or $\sigma_{ij} = Uniform(\bar{S}_t)$ for any $j \in G_i^{out}$. This ensures that all information sets are reached with positive probability under either strategy, thereby bypassing issues arising from off-path beliefs (following a sender's strategy choice). The restriction rules out strategies where a receiver can truthfully reveal some signals while babble for others.Using lemma 1, I can characterise the conditions for truthful communication between neighbours. This allows me to characterise the optimal truthful network: the subgraph $G^{truth} \subseteq G$ where $ij \in G^{truth}$ if and only if i communicates truthfully to receiver j.

Communication stage The sender in our model can employ different communication strategies to different neighbours. Since he cares only about his neighbours (and not the entire network), the neighbours differ only in terms of their preference bias. Consequently all neighbours with the same bias receive the same message from the sender. The optimal communication under the cheap talk protocol is characterised in the following proposition

Proposition 1. *Fix an information structure (S_t, π) and an active receiver i in communication round τ. Under the cheap talk protocol, receiver i truthfully communicates every signal $s \in \bar{S}_t$ to his neighbour $j \in G_i^{out}$ if and only if*

$$2|b_j - b_i| \leq \min\{\mu(\omega = 1|s) - \mu(\omega = 1|s') \geq 0 : \ s, s' \in S_t \} \tag{3}$$

Proposition 1 formalises the intuition that truthful communication between any pair of receivers occur if and only if their preference divergence (measured in terms of the bias magnitude) is low enough. The threshold for truthful communication (the right hand side of the inequality) depends on the information structure and is thus controlled by the sender. First note that when the sender and receiver have the same bias, there will be truthful communication between them irrespective of the information structure (the left-hand side of the inequality is 0). This ensures that both the receivers observe identical history after the communication phase. By lemma 1, they take the same action.[13] Suppose the sender is unbiased and the receiver is biased. Consider two

Figure 6: Incentives of a receiver to transmit signals to her neighbour.

signals s and s' from S_t with $\mu(\omega = 1|s) > \mu(\omega = 1|s')$. Under truthful communication, the sender has to transmit both s and s' truthfully. Upon receiving s, the sender's optimal action is $\mu(\omega = 1|s)$. Under truthful communication, the receiver believes any message that the sender sends as true. If the sender truthfully reveals the signal s to the receiver, the latter chooses $\mu(\omega = 1|s) + b$. The distance between the actions of the two players is b (see the green line in figure 6).[14] If the sender deviates to a signal s', the receiver chooses $\mu(\omega = 1|s') + b$ (since the receiver believes the signal was actually s' under truthful communication). In this case, the distance between the action chosen by the two players is $\mu(\omega = 1|s) - \mu(\omega = 1|s') - b$ (the red line in figure 6). The sender wants others to take an action that is closer to his bliss point. Thus the sender prefers to transmit s over s' iff the distance between the actions upon sending s is smaller than by deviating to s'. In other words, $2b < [\mu(\omega = 1|s) - \mu(\omega = 1|s')]$. Similarly, the sender prefers to transmit s' truthfully over deviating to s if and only if $2b > -[\mu(\omega = 1|s) - \mu(\omega = 1|s')]$. Combining the two, the sender engages in truthful communication for signal pair s, s' with the biased receiver if and only if $2|b| < [\mu(\omega = 1|s) - \mu(\omega = 1|s')]$. Repeating this for all possible signal pair yields proposition 1.

Information design stage The construction of the optimal information structure involves a couple of steps. First I show that the sender can without loss of generality restrict to binary signal spaces. The argument is now fairly standard in the information design literature. With this restriction in place, Proposition 1 and lemma 1 can be used to characterise the sender's payoff. Finally I can use concavification arguments to construct the optimal signals.

From the sender's perspective, the effective action set is binary - he is indifferent

[13]Note that since $|T| = 1$, s is the only information that i can potentially receive through the communication phase.

[14]Here distance refers to the usual metric on $\mathbb{R}$: $d(x, y) = |x - y|$ for x, y in $\mathbb{R}$

between all actions higher (lower) than κ. Any signal realisation s induces an action $a_i^*(s) = argmax_{a \in A}\mathbb{E}[u_i(a,\omega)]$ which can give the sender either a payoff of 1 or 0. Thus the sender can pool all the signals that induce a receiver to take an action $a \geq \kappa$ and send a signal that recommends a "high" action and a signal that recommends a "low" action. The logic is now fairly standard in the information design literature (refer to Lemma 1 of Chan, Gupta, Li, and Wang (2019)). Restriction to binary signal spaces $S_t = \{l, h\}$ vastly simplifies the computation of the sender's payoff. Without loss I assume that $\mu(\omega = 1|h) > \mu(\omega = 1|l)$. To state the optimisation problem of the sender, first define the function $\tilde{v} : \Delta(\Omega) \to \mathbb{R}$ which describes the payoff that the sender obtains for each possible posterior that can be induced by the sender. Each signal realisation s from (S_t, π) induces a posterior distribution $\mathbf{p} \in \Delta(\Omega)$, which is sufficient to predict the receivers' optimal actions. Hence (S_t, π) induces a distribution λ over the set of possible posteriors. That is, $\lambda \in \Delta(\Delta(\Omega))$ yields an expected payoff of $\mathbb{E}_\lambda[\tilde{v}(\mathbf{p})]$ to the sender. Applying corollary 1 of Kamenica and Gentzkow (2011), I can write the sender's optimisation problem as follows:

$$\max_{\psi \in \Delta(\chi)} \sum_{(S_t,\pi)\in\chi} \psi\Big((S_t,\pi)\Big)\Big[\max_{\lambda(S_t,\pi)\in\Delta(\Delta(\Omega))} \mathbb{E}_{\lambda(S_t,\pi)}[\tilde{v}(\mathbf{p})]\Big] \tag{4}$$

$$\text{s.t. } \mathbb{E}_{\lambda(S_t,\pi)}[\mathbf{p}] = \mu_0 \tag{5}$$

This optimisation problem can be solved in two steps. The key idea is to partition the constraint space χ into n components: $\chi = \{(S_1, \pi); (S_2, \pi); \ldots; (S_n, \pi)\}$ where (S_t, π) is the information structure with t seed. In step 1, I find the optimal $\lambda(S_t, \pi)$ for each possible choice of target t. This is identical to a standard Bayesian persuasion model with the sender's payoff adapted to our setting. Using this payoff, I can determine the optimal target for the sender.

Let us characterise the payoff function $\tilde{v}$ as a function of the receivers' optimal actions. Consider any feasible information structure $(S_t, \pi) \in \chi$. The Bayes' plausibility constraint 5 requires that $\mu(\omega = 1|l) < \mu_0(\omega = 1) < \mu(\omega = 1|h)$. The l signal will lead to an action $a_i < \kappa$ for any receiver in the network, irrespective of the communication strategy profile. Hence $A_\kappa = \emptyset$ and the payoff to the sender is $v(\emptyset) = 0$. Consequently a receiver can be persuaded to take an action $a_i \geq \kappa$ only upon receiving a h signal. Hence the sender only cares about the diffusion of h signal through the network. Lemma **??** formally states how the actions and communication strategy of the receivers depend on

(S_t, π).

Lemma 2. *Consider an information structure (S_t, π) with $S_t = \{l, h\}$.*

- ***High action**:*
  a receiver i chooses $a_i \geq \kappa$ if and only if $\frac{\pi(h|0)}{\pi(h|1)} \leq \frac{1-\kappa+b_i}{\kappa-b_i}\frac{\mu_0(\omega=1)}{\mu_0(\omega=0)}$.
  - *The cutoff for a biased receiver is $\pi^{**} \equiv \frac{1-\kappa+b}{\kappa-b}\frac{\mu_0(\omega=1)}{\mu_0(\omega=0)}$ whereas the cutoff for an unbiased receiver is $\pi^{*} \equiv \frac{1-\kappa}{\kappa}\frac{\mu_0(\omega=1)}{\mu_0(\omega=0)}$.*
  - *The difference in threshold $(\pi^{**} - \pi^{*})$ is positive and increasing in b.*
- ***Truthful Communication**:*
  A sender i in round τ truthfully communicates both the high (h) and low (l) signal to his neighbour $j \in G_i^{out}$ if and only if

$$2|b_j - b_i| \leq \frac{\mu_0(\omega=1)}{\mu_0(\omega=0)}\left[\frac{\pi(h|\omega=1)}{\pi(h|\omega=0)} - \frac{\pi(l|\omega=1)}{\pi(l|\omega=0)}\right]$$

Any signal h need not lead the receivers to choose the high action. It is necessary for the information structure to induce a likelihood ratio of h signal realisation that is low enough. Bayes' rule then implies that the posterior induced by the h signal is high enough. The fact that optimal action is an increasing in the induced posterior probability of $\omega = 1$ implies that the receivers follow a simple threshold rule: choose the "high" action if and only if the induced posterior probability of $\omega = 1$ exceeds a threshold value. Lemma **??** shows that this cutoff is higher for a biased receiver (π^{**}) compared to an unbiased receiver (π^{*}). It is in this sense that the sender finds it relatively easier to persuade the biased receivers to choose $a_i \geq \kappa$. The reason behind the threshold π^{**} to be increasing in b is intuitive: a higher b implies that the preferences are more aligned towards the sender's objective, and hence is easier to convince him to choose the high action.

The condition for truthful communication is more complicated and makes the exercise non-trivial. Lemma **??** shows that this condition cannot be pinned down by the posteriors induced by the signals. Instead one needs knowledge of the entire information structure to compute the receivers' optimal actions and thus the sender's payoff. Hence it is generally not possible to read off the optimal signals using the concavification procedure as in Kamenica and Gentzkow (2011).[15] Under the binary

[15] Bloedel and Segal (2019) also faces similar issue in a different context.

action setup however, I can bypass this issue by further simplifying the the structure of optimal information structures in any equilibrium. It is straightforward to see that any optimal information structure must have $\pi(h|1) = 1$ irrespective the target. The logic is exactly the same as in the jury example of Kamenica and Gentzkow (2011). The receivers' optimal action following a h or l signal is given by

$$\begin{aligned} a_i^*(h) &= \frac{1}{1 + \frac{\mu_0(\omega=0)}{\mu_0(\omega=1)} \frac{\pi(h|\omega=0)}{\pi(h|\omega=1)}} + b_i \\ a_i^*(l) &= \frac{1}{1 + \frac{\mu_0(\omega=0)}{\mu_0(\omega=1)} \frac{\pi(l|\omega=0)}{\pi(l|\omega=1)}} + b_i \\ &= \frac{1}{1 + \frac{\mu_0(\omega=0)}{\mu_0(\omega=1)} \frac{1-\pi(h|\omega=0)}{1-\pi(h|\omega=1)}} + b_i \end{aligned}$$

Thus $a_i^*(h)$ is increasing in $\pi(h|\omega = 1)$ and $a_i^*(l)$ is increasing in $\pi(h|\omega = 0)$. From the sender's perspective, he wants to increase both $\pi(h|\omega = 1)$ and $\pi(h|\omega = 0)$ as high as possible. Since the information structure must be informative (i.e. $\pi(h|\omega = 1) > \pi(h|\omega = 0)$), we must have that $\pi(h|\omega = 1) = 1$. However, $\pi(h|\omega = 0) = 1$ need not be optimal - since this renders the information structure completely uninformative. Consequently, any optimal information structure has the following form Thus a low

		S_t^*	
		l	h
ω	0	$1-\pi(h\|0)$	$\pi(h\|0)$
	1	0	1

Table 3: Structure of the optimal information structure under binary states

signal l completely reveals the state to be $\omega = 0$ and the induced posterior must be degenerate: $\mu(\omega = 1|l) = 0$. The precision of information is then uniquely determined by the sender's choice of $\pi(h|0)$. Notice that this simplifies the condition for truthful communication to

$$2|b_j - b_i| \leq \frac{\mu_0(\omega = 1)}{\mu_0(\omega = 0)} \frac{1}{\pi(h|0)} \tag{6}$$

I can now combine proposition 1, lemma 1, and lemma 2 to provide a simple characterisation of the receivers' strategies in terms of $\pi(h|0)$.

Lemma 3. *Consider an information structure (S_t, π) with $\pi(h|1) = 1$. Then the optimal strategies of the receivers is as follows:*

- *When $\pi(h|0) \leq \bar{\pi}$: all receivers choose the high action, unbiased receivers transmit s to biased receivers.*
- *When $\bar{\pi} < \pi(h|0) \leq \pi^*$: all receivers choose the high action, unbiased receivers block s to biased receivers.*
- *When $\pi^* < \pi(h|0) \leq \pi^{**}$: only biased receivers choose the high action, unbiased receivers block s to biased receivers.*

To construct the sender's payoff, we must obtain the equilibrium truthful networks for each information structure: the subgraph of G where receivers truthfully reveal all signals to their neighbours. Let us define $\bar{v} : [0,1] \to \mathbb{R}$ to be the payoff that the sender obtains for any posterior probability of $\omega = 1$. Suppose the sender targets an unbiased receiver t in the network. For any posterior $p \in [0, \bar{\mu}]$,[16] the unbiased receiver babbles to the biased receivers (who thus take the low action). In this case, the equilibrium truthful network involves the subgraph where all paths start from the seed, and involves only unbiased receivers. That is $G_t^{truth} = \{j \in N : \exists (t = i_1, i_2, \ldots, i_K = j)$ such that $i_k i_{k+1} \in G$ and $b_{i_k} = b_{i_{k+1}}$ for all $k = 1, 2, \ldots, K-1\}$. Let us look at the actions that the receivers take. When $p \in [0, \mu^*)$, all receivers in the truthful equilibrium network take the low action irrespective of the communication strategies. Hence $\tilde{v}(p) = 0$ in such a case. When $p \in [\mu^*, \bar{\pi})$, all receivers in the truthful equilibrium network choose the high action, giving the sender a payoff of $v(G_t^{truth})$. Consider the case where $p \in [\bar{\mu}, 1]$. When posterior is high enough, all receivers truthfully communicate with each other, i.e. $G_t^{truth} = G_t$. Moreover all receivers in the equilibrium truthful network take the high action. Thus the sender obtains $v(G_t)$. One can write down the payoffs analogously when t is a biased receivers. Hence we can summarise $\bar{v}$ in any optimal information structure as follows

$$\bar{v}(p) = \begin{cases} 0 & \text{if } p \in [0, \mu(t)) \\ v(G_t^{truth}) & \text{if } p \in [\mu(t), 2b) \\ v(G_t) & \text{if } p \in [2b, 1] \end{cases} \tag{7}$$

[16] Here $\bar{\mu}$ is the posterior probability of $\omega = 1$ induced by an optimal information structure with $\pi(h|0) = \bar{\pi}$

where $\mu(t) \in \{\mu^*, \mu^{**}\}$ is receiver t's threshold for taking the "high" action. Since $G_t^{truth} \subseteq G_t$, the sender's payoff $\bar{v}$ is weakly decreasing in p. This formalises the fundamental trade-off that the sender faces: a higher $\pi(h|0)$ increases the probability of h signal realisation. This benefit is countered by decreased diffusion of the h signal (summarised by the $\bar{v}$ function). The sender optimally trades off between these two forces in his optimal strategy.

Despite the issues arising due to the condition for truthful communication, I can use $\bar{v}$ to characterise the sender's payoff using the concavification technique (corollary 2 of Kamenica and Gentzkow (2011)).[17] For the sake of completeness, I'll define the concave closure of a function below

Definition 3. *The concave closure of the function $\tilde{v}(\mathbf{p})$ is defined to be the set*

$$F(\mathbf{p}) \equiv sup\{z : (\mathbf{p}, z) \in co(\tilde{v}(\mathbf{p}))\}$$

where $co(\tilde{v}(\mathbf{p}))$ denotes the convex hull of the graph of $\tilde{v}(\mathbf{p})$.

The maximum value that the sender obtains from a feasible information structure $(S_t, \pi) \in \chi$ is given by $F_{(S_t,\pi)}(\mu_0)$ (Corollary 2 of Kamenica and Gentzkow (2011)). Although $F_{(S_t,\pi)}(\mu_0)$ may be complicated in general, binary states allow us to obtain a simple geometric characterisation for $F_{(S_t,\pi)}(\mu_0)$.

Proposition 2. *Suppose the sender targets receiver $t \in N$ and $\pi(h|0) = 1$. Define $\beta = max\{\frac{v(G_t^{truth})}{\mu(t)}, \frac{v(G_t)}{2b}\}$. Then the maximum payoff that the sender obtains from targeting t is given by*

$$F_{(S_t,\pi)}(\mu_0) = \beta\mu_0(\omega = 1)$$

The optimal information structure involves inducing two posteriors μ^h and μ^l as follows

$$(\mu^l, \mu^h) = \begin{cases} (0, \mu(t)) & if\ \frac{v(G_t^{truth})}{v(G_t)} > \frac{\mu(t)}{2b} \\ \alpha(0, \mu(t)) + [1-\alpha](0, 2b) & if\ \frac{v(G_t^{truth})}{v(G_t)} = \frac{\mu(t)}{2b} \\ (0, 2b) & if\ \frac{v(G_t^{truth})}{v(G_t)} < \frac{\mu(t)}{2b} \end{cases} \quad (8)$$

[17] Note that this works along the equilibrium path where we have pinned down that one signal must be induce a posterior belief (about $\omega = 1$) of 0. The proof of proposition **??** does not depend on the concavification argument.

The intuition for the expression for $F_{(S_t,\pi)}(\mu_0)$ is easy to see in terms of figure 7. Under Assumption 2, the prior $\mu_0(\omega = 1) < \mu^*$. All I need to calculate is the slope of the function $F(\mathbf{p})$ at $\mu_0(\omega = 1)$. This slope is given by β in proposition 2. This simple geometric argument tells us that the value that the sender obtains from (S_t, π) is thus $F_{(S_t,\pi)}(\mu_0) = \frac{\mu_0(\omega=1)}{\mu(t)} v(G_t^{truth})$. The optimal information structure induces two posteriors, $\mu^l(\omega = 1) = 0$ and $\mu^h(\omega = 1) = \mu(t)$. Bayes' rule then pins down the values of π. We have thus found the optimal information structure following each target (S_t^*, π^*). Let $\bar{\chi}_1 = \{(S_t^*, \pi^*)\}_{t\in N}$ be the set of such optimal information structures. (Note that this is a set whose cardinality is n). Replacing $F_{(S_t,\pi)}(\mu_0)$ in problem 4 we can write the targeting problem of the sender as a simple linear programming problem:

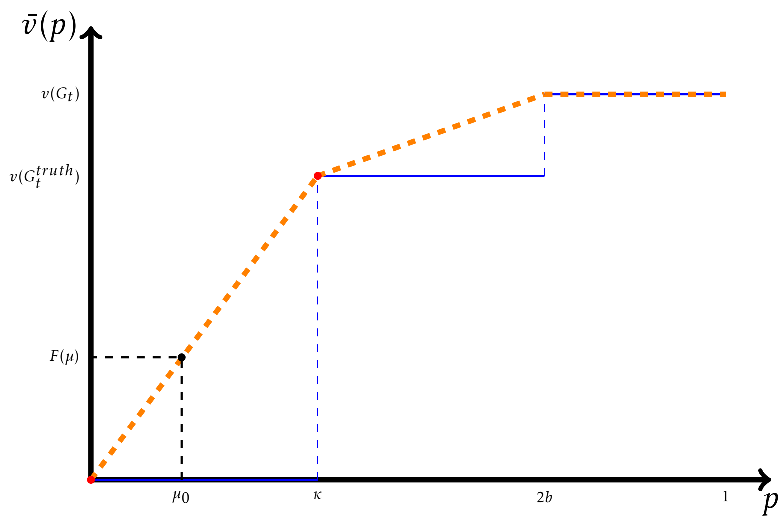

Figure 7: Concave closure $\tilde{v}$ for the receiver's expected payoff.

$$\max_{\psi\in\Delta(\bar{\chi}_1)} \sum_{(S_t^*,\pi^*)\in\chi} \psi\Big((S_t^*, \pi^*)\Big) F_{(S_t^*,\pi^*)}(\mu_0) \tag{9}$$

The solution to problem 9 has a bang-bang nature characterised in the next proposition

Proposition 3. *Define* $\chi^* = \{(S_t, \pi) \in \chi : F_{(S_t^*,\pi^*))}(\mu_0) \geq F_{\overline{(S_t^*,\pi^*)}}(\mu_0) \ \forall \ \overline{(S_t^*, \pi^*)} \in \chi\}$. *The optimisation problem 9 is well-defined and has the following solution:*

$$\psi^*((S_t, \pi)) = \begin{cases} \frac{1}{|\chi^*|} & \text{if } (S_t, \pi) \in \chi^* \\ 0 & o.w. \end{cases} \tag{10}$$

While the intuition of proposition 3 is straightforward, it does not provide a clear insight into the role of the network structure. Notice that the network structure impacts the sender's payoff (and hence his decision) through its dependence on G_t^{truth} and G_t. To this end, I define a measure that characterises G_t^{truth} in terms of σ and primitives of the network structure. From the sender's perspective, a target's relevance depends on his ability to diffuse information and persuade others to choose the high action. This suggests an intuitive way to measure the importance of each receiver in the network. Consider a target t and a non-target j. Let w be a path of length l_w from t to j and W_{tj} be the set of all such paths. Information diffuses from t to j if and only if every receiver along w communicates truthfully to the next receiver along the path, i.e. if w is a path in the truthful communication network G_t^{truth}. We can define the importance of the receivers as follows

Definition 4. *The influence of receiver t in the network G is the expected number of receivers who choose the high action when receiver t is seeded with an information structure* (S_t, π)

$$C_i(G, \{S, \pi\}) = \sum_{j \in N} Prob\{a_j \geq \kappa\}$$

where

$$Prob\{a_j \geq \kappa\} = \sum_{s \in S} Pr\{s\} \Big[\sum_{w \in W_{tj}} \prod_{k=1}^{l_w - 1} \sigma_{i_k i_{k+1}}(s) \mathbf{1}_{a_j \geq \kappa} \Big]$$

is determined endogenously by the strategies of the receivers and the sender.

"Influence" of receiver t is determined by the strategies of the receivers and the sender, thereby making it dependent on parameters other than the network structure. Hence it is technically not a centrality measure of the nodes in the network G. This measure reduces to the diffusion centrality defined by Banerjee, Chandrasekhar, Duflo, and Jackson (2014), Banerjee et al. (2013) under certain conditions. One can interpret receiver t's influence as follows. Suppose seed t is observes a signal s from (S_t, π). Consider any other receiver j and let w be a path starting from i and ending in j. receiver j receives the s if and only if every receiver along w decides to transmit the message when it is their turn. Hence the signal s reaches receiver j from i with probability $\prod_{k=0}^{l(w)-1} \sigma_{i_k i_{k+1}}(s)$. The conditional probability (conditional on signal s realised) of receiver

j choosing high action when i is the seed is thus $\prod_{k=0}^{l(w)-1} \sigma_{i_k i_{k+1}}(s)\mathbf{1}_{a_j \geq \kappa}$.

Since a receiver with higher influence can induce a larger number of receivers to choose the high action on average, sender wants choose a seed with the highest centrality measure. This is formally stated in the next result which characterises the sender's choice of information structure in terms of influence.

Theorem 1. *Consider the optimal information structures* (S_t^*, π^*) *that the sender chooses for any possible seed* $t \in \{1, 2, \ldots, n\}$. *The optimal seeding strategy* ψ^* *of the sender is the characterised by the following equivalence relation*

$$\psi^*((S_i, \pi)) > 0 \Leftrightarrow C_i(G, (S_i^*, \pi^*) \geq C_j(G, (S_j^*, \pi^*) \; \forall \; j \neq i, \; i, j, \in N \tag{11}$$

The interpretation is straightforward. Intuitively, diffusion centrality of a receiver i captures the expected number of receivers who receive a piece of information when i is seeded with that information. In our context, the piece of information is the signal s realised from the information structure. Under cheap talk, all receivers who receive a truthful signal takes the same action: either the truthful communication network comprises of receivers with the same bias, or it includes all receivers in G_t (but the posterior is high enough in such a case so all receivers take the high action) The only difference between influence and diffusion centrality here stems from the fact that receivers receive the h signal with positive probability even when all his neighbour babbles to him. Since receivers choose the low action under babbling, the expected number of people choosing the high action is different from the expected number of people receiving the h signal. Hence influence and diffusion centrality differs in this case.

Example 2. *Suppose there are 3 receivers,* $N = \{1, 2, 3\}$, *where receiver 1 is unbiased, 2 and 3 are biased. The receivers are connected in a line graph as follows: The magnitude of the*

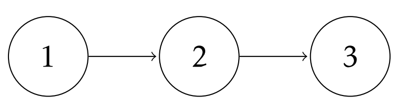

Figure 8: Line network with receiver 1 unbiased, 2 and 3 biased.

bias is $b = 0.3$, *the common prior is* $\mu(\omega = 1) = 0.1$, *and* $\kappa = 0.5$. *The sender can choose only 1 seed but can send any information structure* (S_t, π) *to the seed. The optimal actions of the receivers are given by*

- *Unbiased: Chooses* $a_1^* = 0.1$ *upon observing nothing,* $a_1^* = \mu(\omega = 1|s)$ *upon observing* s.
- *Biased: Chooses* $a_i^* = 0.4$ *upon observing nothing,* $a_i^* = \mu(\omega = 1|s) + 0.3$ *upon observing* s.

Note that the thresholds for taking the high action are 0.2 *and* 0.5 *for the biased and unbiased receivers respectively. We know that the posterior induced after a* l *signal in any optimal information structure is characterised* $\mu(\omega = 1|s) = 0$. *This allows us to determine the truthful communication network under any optimal information structure.*

- $\mu(\omega = 1|h) < 0.6$*:* $G_1^{truth} = \{1\}$, $G_2^{truth} = \{2,3\}$, $G_3^{truth} = \{3\}$.
- $\mu(\omega = 1|h) \geq 0.6$*:* $G_1^{truth} = \{1,2,3\}$, $G_2^{truth} = \{2,3\}$, $G_3^{truth} = \{3\}$.

I can now use proposition 2 to characterise the highest payoff that the sender can achieve upon seeding a receiver:

- *Seed is 1: Then* $\beta = max\{\frac{1}{0.5}, \frac{3}{0.6}\} = 5$*. Hence* $F_{(S_1,\pi)}(0.1) = 0.1 * 5 = 0.5$.
- *Seed is 2: Then* $\beta = \frac{2}{0.2} = 10$*. Hence* $F_{(S,\pi)(2)}(0.1) = 0.1 \times 10 = 1$.
- *Seed is 3: Then* $\beta = \frac{1}{0.2} = 5$*. Hence* $F_{(S,\pi)(3)}(0.1) = 0.1 \times 5 = 0.5$.

Since the sender obtains the highest payoff upon seeding receiver 2, his optimal targeting strategy is to choose receiver 2 with probability 1. The sender thus targets receiver 2 and sends him the following information structure:

		S^*: 0	S^*: 1
θ	0	$\frac{5}{9}$	$\frac{4}{9}$
	1	0	1

It is easy to see how this model relates to the standard bayesian persuasion literature. The current literature on bayesian persuasion (and information design in general) implicitly assumes extreme communication structures in their setup. One extreme case is obvious. Suppose the network is empty. In this case, a sender who can target only one receiver plays a single receiver bayesian persuasion model as in Kamenica and Gentzkow (2011). In this case the sender always targets the biased receiver.[18]. Consider the other extreme where each receiver is connected to every other receiver in the network. I show that such a setup coincides with a public persuasion model as in Arieli and Babichenko (2016).

[18]If we allow for multiple targets, an empty network basically corresponds to models of private bayesian persuasion [see Arieli and Babichenko (2016),Wang (2013), Chan et al. (2019)]

Lemma 4. *Suppose the network G is strongly connected, and all receivers are unbiased. The optimal strategy of the sender is as follows*

- ***Targeting****: indifferent between any receiver, i.e. any $\psi \in \Delta(\chi)$ is optimal.*
- ***Optimal Information****: the optimal information involves $S_t = \{l, h\}$ with $\pi(h|0) = \pi^*$, $\pi(h|1) = 1$.*

Every receiver chooses the high action yielding the sender a payoff of

$$\left[\mu(\omega = 1) + \mu(\omega = 0)\pi^*\right] v(N)$$

Since the network is strongly connected, there is a (directed) path from the seed $t \in T$ to any other receiver $j \in N \setminus \{t\}$. This implies that a signal that the seed observes reaches every other receiver in the network. It is as if the sender constructs an information structure whose signal realisation is observed by every receiver, and hence equivalent to a public persuasion model à la Arieli and Babichenko (2016) (Theorem 4). The sender gets the same payoff $v(N)$ irrespective of who he targets.

The next example shows that when the receivers are connected but not strongly so, the payoff need not coincide with Theorem 4 of Arieli and Babichenko (2016). This is because the signal need not reach every receiver under such a network structure.

Example 3. *Suppose $N = \{1, 2, 3\}$ and consider the line graph $g\{12, 32\}$. Assume that every receiver in the network are biased. The preferences and other parameters are exactly as in the illustration above (with $b = 0.1$). The payoff that the sender gets upon targeting 1 or 3 is*

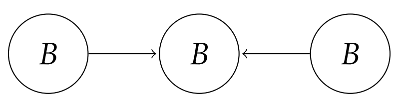

FIGURE 9: A network may be connected, but not strongly connected.

: $0.67 * \frac{0.2}{0.4} = 0.33$. *Similarly, the payoff from targeting 2 is* $0.33 * 0.5 = 0.16$. *The optimal target is player 1 or 3, which is different from $v(3)$ as we have in lemma 4. the optimal information structure is given by*

		S	
		0	1
θ	0	0.25	0.75
	1	0	1

5 Comparative statics

The analysis of Section 4 highlights the fundamental trade-off that the sender faces. This is reflected in the optimal policies characterised in Proposition 2 and Theorem 1. A natural follow-up to this would be to study how these policies depend on the network structure and the composition of biased agents in the network. We focus on these issues in this section where three comparative static analysis are discussed. First, I show that as the bias magnitude b decreases, the sender may find it profitable to switch targets and choose a more well-connected receiver. I obtain similar results by increasing the proportion of biased receivers in the network. Because of this target switching, I study the impact of expanding the network structure on the sender's payoff when all receivers are homogeneous.

5.1 Polarisation: bias vs network structure

Theorem 3 tells us that the sender's optimal seed set includes receivers only with the highest influence in the network. Influence of a receiver is in general a complicated function of the bias and the network position. This makes it difficult to study how the sender trades off between these two aspects of the receivers' heterogeneity. Consequently I restrict analysis to the case where the network G is connected. Hence all receivers receive the signal irrespective of the seed chosen by the sender. Hence the optimal information structure depends on the proportion of biased and unbiased receivers in the network. I interpret the magnitude of the bias b as the extent of polarisation among the receivers in the society. What happens if the network G is fixed and decrease b?[19] The next result shows that decrease in polarisation leads to more informative news and larger spread of information in the network.

Lemma 5. *Suppose G is strongly connected. Then*

- *sender's optimal (expected) payoff is continuous and weakly increasing in $b \in [0, \kappa - \mu_0(\omega = 1)]$.*
- *optimal signal structure is weakly more informative as b decreases, and strictly more informative in the as long as $b \in \left[\bar{b}, \kappa - \mu_0(\omega = 1)\right]$ with $\bar{b} = \kappa\left[1 - \frac{v(B)}{v(N)}\right]$*

[19] I maintain that Assumption 3 still holds, and thus the change in bias cannot be high/low enough to change the order of the threshold values.

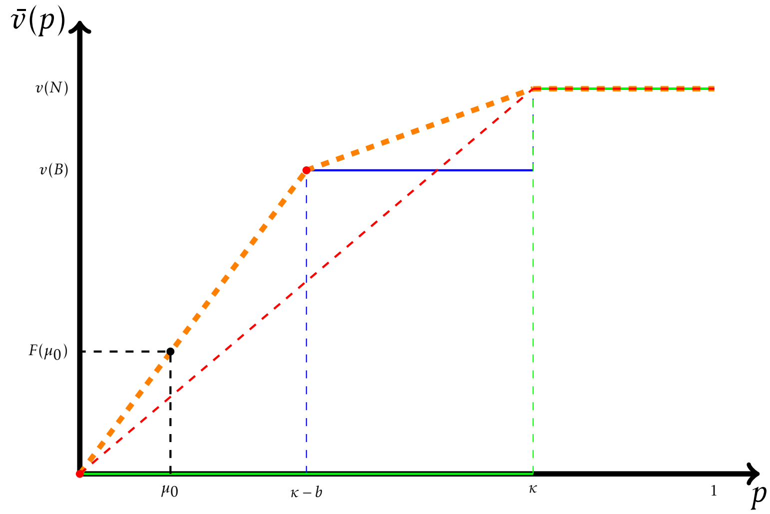

FIGURE 10: Effect of changing b on information and sender payoff

A diagrammatic exposition provides a nice way to understand the forces at hand. Consider figure 10. It is clear that the bias and network structure affect the sender's payoff through different channels: bias affects the threshold (horizontal shifts), whereas the network structure affects the sender's payoff through $\bar{v}$ (vertical shifts). A quick glance at figure 10 tells us that the threshold $\kappa - b$ to shift to the right, keeping the values of $\bar{v}$ unchanged. The preference of the two types of receivers are relatively more aligned as b decreases Hence the biased receiver requires more accurate information to be persuaded to take the "high" action. This reduces the benefit of targeting a biased receiver. Beyond a threshold, the sender finds it optimal to choose an information structure that persuades both the biased and unbiased receivers.

Using lemma 9, the payoff from the orange concave closure in figure 10 is $\mu_0(\omega = 1)\frac{v(B)}{\kappa - b}$ while that from the red concave closure is $\mu_0(\omega = 1)\frac{v(N)}{\kappa}$. The initial bias in figure 10 is high enough so that the sender only conforms to the biased receivers. As b decreases, the payoff from targeting an unbiased receiver increases. When b is low enough,[20], the benefit of targeting all receivers in the network is high enough to overcome the cost of sending a more precise information structure.

[20]More precisely, when $b < \kappa - \frac{\kappa v(N)}{v(B)}$

5.2 Pure network effect

To study the impact of the network structure on the sender's payoff, I assume that all the receivers are homogeneous. It is easy to verify that the optimal information structure when all the receivers are unbiased involves $\pi(h|0) = \pi^*$ irrespective of the target. Hence the network structure solely determines the number of receivers the sender can reach from choosing a particular target $t \in N$. Since the communication strategies are trivial, this setup can be interpreted as a pure information spillover model where information flows automatically through links in the network. How is the sender's payoff affected when the network structure changes? To answer this question, I first need to define what "changing the network structure" means in this context. From the sender's perspective, he cares how far each receiver can spread information when chosen as a seed. This motivates the following definition

Definition 5. *A network G' is more reachable than G if and only if $G_i \subseteq G'_i$ for all $i \in N$ and $G_i \subset G'_i$ for at-least one $i \in N$.*

It is worth noting that if G is a subgraph of G', then G' is trivially more reachable than G. The converse, however, is not true. Consider the following example with $N = \{1,2,3\}$. Suppose $G = \{(1,3),(2,3)\}$ and $G' = \{(1,3),(2,1)\}$ (see figure 11). It is clear that $G_i \subseteq G'_i$ for all $i \in N$ and $G_2 \subset G'_2$. Hence G' is more reachable than G. However, G and G' cannot be ranked using set inclusion, and hence one is not a subgraph of the other. The next result shows that the sender always prefers a network that is more

FIGURE 11: G' is more reachable than G, but is not a sub-graph of G.

reachable. This is fairly intuitive and follows from the discussion in section 3.1. With preference homogeneity, a more reachable network implies that a larger set of receivers can potentially choose the high action. This provides a higher payoff to the sender. Let V be the optimal (ex-ante) payoff of the sender

Theorem 2. *Suppose $b_i = b_j$ for all $i, j \in N$, and let G, G' be two networks of n receivers.*

Then

$$G' \text{ is more reachable than } G \implies V(G) \leq V(G')$$

Suppose G' is more reachable than G. For any seed $t \in T$, I must have $G_t \subseteq G'_t$ by definition. Since all information is transmitted to every receiver, every receiver in G_t (equivalently in G'_t) receives the favourable signal when t is the seed. Thus the number of receivers choosing the high action is $G_t \subseteq G'_t$ for all $t \in T$, giving the sender a payoff of $v(G_t) \leq v(G'_t)$. Since the accuracy of the information structure is independent of G in this setup, it follows that $V(G) \leq V(G')$. The converse is not true. To see this consider the following example.

Example 4. Suppose $N = \{1,2,3\}$. Consider the networks $G = \{(1,2),(3,2)\}$ and $G' = \{(1,2),(2,3)\}$. A simple application of lemma 9 shows that $V(G) = \frac{\mu(\omega=1)}{\kappa}v(G_1)$ and $V(G') = \frac{\mu(\omega=1)}{\kappa}v(N)$. Since $G_1 = \{1,2\} \subset N$, I have that $V(G) \leq V(G')$. However it is easy to see that neither G nor G' is more reachable than the other: $G'_3 \subset G_3$ but $G'_2 \supset G_2$.

An immediate corollary to theorem 2 is that the sender can obtain the highest payoff when the network is complete.

Corollary 1. *For any network G with $\emptyset \subset G \subset N^2$, it must be* $0 < V(G) \leq V(N^2)$.

This highlights the primary driving force that the sender faces. He wants every receiver in the network to take the high action, but is constrained to provide information "locally" to a small portion of the network. He depends on the network to induce a global spillover of this localised information. In a complete network N^2 each receiver i has a directed link to rest of the $n-1$ receivers, i.e. $G_i = N \; \forall \; i \in N$. Hence the sender receives a payoff of $v(N^2)$ from the realised h signal. For any other network G, we have $G_i \subseteq N$, which by theorem 2 implies that $V(G) \leq V(N^2)$. In fact, the complete network N^2 is the most reachable network, while the empty network is the least reachable network.

5.3 Increasing proportion of biased receivers

The last case we study is the effect of increasing the proportion of biased receivers in the network. Fix the magnitude of the bias b and the network G and increase $|B|$. We maintain the assumption that G is strongly connected and suppose an unbiased receiver

i in the network is replaced by a biased receiver. Thus the number of receivers in the network is remains fixed at $|N|$, but the proportion of biased receivers $|B|$ increases. The next result shows that the sender's payoff increases and informativeness of the of the optimal signal structure decreases as we increase the proportion of biased receivers.

Lemma 6. *Suppsoe the network G is strongly connected and let $\gamma = \frac{|B|}{|N|}$ be the proportion of biased receivers. Then the optimal signal structure is more informative as γ decreases, and is given by*

$$\pi(h|0) = \begin{cases} \pi^{**} & \gamma \in [1 - \frac{b}{\kappa}, 1] \\ \pi^{*} & \gamma \in [0, 1 - \frac{b}{\kappa}] \end{cases}$$

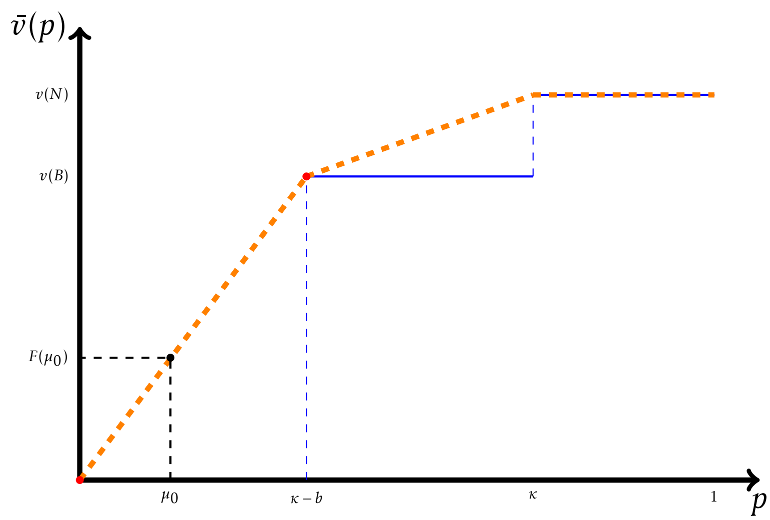

Figure 12: Effect of increasing the proportion of biased receivers on information and sender payoff.

Consider the concave closure in Figure 12. As γ increases, the horizontal line at $v(B)$ shifts parallel upwards, while everything else remains constant.[21] If the shift is high enough, the the benefit of spreading information is not high enough. Therefore the sender only conforms to the biased receivers in the network and chooses an information structure just enough to persuade them.

The interpretation is pretty straightforward. If there are few biased receivers in G, the sender is better off in maximising the spread of the signal -the precision is such

[21]Note that $v(N)$ remains unchanged since the total number of receivers in G remains unchanged by the switch.

that the unbiased receivers is just willing to transmit the signals to the biased receivers. On the other extreme, suppose there are sufficiently high biased receivers. Then the sender forgoes maximal spread of information and provides an information that is just precise enough to persuade only the biased receivers. Lemma 6 provides an alternative rationale as to why politicians provide services or information to members with the same political affiliations. In my model, the sender (or the politician) doesn't obtain any additional payoff from seeding a biased receiver: thus clientelism is not the reason for targeting biased receivers. The sender conforms to the biased receivers in equilibrium simply because they are easier to convince and sufficiently large in number in the model. There is some recent empirical evidence of such targeting by local governments and microcredit institution.[22]

6 Conclusion

I present a model of multi-receiver bayesian persuasion where the sender is constrained to choose only a subset of receivers and the latter can communicate privately with each other within a social network. Such a setting covers situations that include lobbying, political campaigning, viral marketing, etc. I show that in such a setting the sender trades off between maximising spread of information with catering to receivers biased towards him. The optimal targeting strategy of the sender involves choosing a receiver with the highest influence measure - a novel generalisation of diffusion centrality. When all receivers have the same bias, influence reduces exactly to diffusion centrality., thereby providing a micro-foundation for diffusion centrality. I also show that as the polarisation in the society increases, the sender finds it optimal to cater only to the biased receivers at the cost of lower spread of information. This gives a competing story to clientelism as a rationale to the common observation that services are provides disproportionately to members with the same political or social affiliation.

There are a number of ways to extend this paper for future research. Some of the generalisations are relatively straightforward and preserves the qualitative results of the paper. One can generalise the payoffs to include arbitrary biases and state dependent sender preferences. Multiple states can also be incorporated with slight modifications. However. incorporating multiple targets in a general setup is difficult due to the

[22] I am indebted to Dilip Mookherjee for this insight.

complications arising out of bayesian updating in networks. A tractable model for such full bayesian receivers in a social network setting is an important area for further investigation. The model in its current form has some testable implications which can be verified empirically. One such implication of interest is the fact that the sender tends to cater to only biased receivers when the polarisation in the society is too high or there are sufficiently large. Using text data and leveraging social network data from sites like Twitter is a good starting point to test the patterns of information spread predicted by this paper. The analysis also serves as benchmark to study the propagation of "fake news" within the society. Who is more likely to share "fake news" and with whom? How do receivers process information when faced with "fake news", and how does this depend on the network structure? Such analysis may shed light on the increasingly difficult task to stem the rapid flow of falsified information on social media sites. These are extremely pertinent questions to analyse, especially in the digital age, and we leave these as fruitful avenues of future research.

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7 Appendix

7.1 Networks and Centrality

The following are some definitions that have been used in the paper:

- The *(out-) neighborhood* of agent i is the set of agents in N to whom i can communicate with directly, i.e. $G_i^{out} = \{j \in N : (i,j) \in G\}$.
- The *(in-)neighborhood* of agent i is the set of agents in N who can communicate with i directly: $G_i^{in} = \{j \in N : (j,i) \in G\}$.
- A *path* from agent i to agent j in G is a sequence of distinct agents $(i_1, i_2, \ldots, i_K)$ such that $i = i_1, i_K = j$ and $i_k i_{k+1} \in G$ for all $k = 1, 2, \ldots, K-1$.
- The set of agents $C \subseteq N$ is a *component* of G if there is a path from i to j for all distinct $i, j \in C$ and $N_i \subseteq C$ for all $i \in C$. The set of components of g is denoted by $\mathcal{C}(G)$.
- A network G is *strongly connected* if there exists a (directed) path w between any two pairs of agents $i, j \in N$.
- A network G is connected if the underlying undirected graph $\hat{G}$ is connected.

Lemma 7. *G is strongly* connected $\implies$ *G is connected.*

Proof. Let G be any strongly connected graph. Then for all $1, j \in N$, there exists a directed path $w \equiv (i_1, i_2, \ldots, i_K)$ with $i_1 = i, i_K = j$ and $(i_k, i_{k+1}) \in G$ for $k = 1, 2, \ldots, K-1$. This implies that there exists an undirected path $\hat{w}$ from any agent i to j in the network. (just remove the arrow heads from w). Hence G is connected. □

The converse is not true. Consider the following graph:

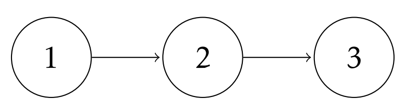

Figure 13: Directed line graph

This graph is connected, but not strongly so.

The following simple algorithm identifies the subgraph along which the signal s percolates given that the Designer targets agent $t \in T$.

- **Input:** graph G, target $t \in T$, π, parameters κ, μ, b

- **Step 1:** If $0 \leq \pi(1|0) \leq \bar{\pi}$:
  - **Step 1a:** return $G^* = G$ and N. Stop.
- **Step 2:** else:
  - **Step 2a:** Start with $t \in T$.
    Let $G_t = \{j \in N : \exists \text{a directed path from } t \text{ to } j.\}$. Let $G^1 = G(G_t \cup t)$ be the subgraph induced by the vertices in G_t and t.
  - **Step 2b:** Delete all links $(i,j) \in G^1$ such that $b_i < b_j$. Let the resulting subgraph be G^*
- **Output**: return G^* and $G_t^1 \cup \{t\}$. Stop.

The idea of the algorithm is fairly simple. If agent t is a target, then the signal $s = 1$ can only reach the agents who are connected to t. Consider the subgraph induced by t and G_t. The algorithm then parses out the portions of the subgraph where the agents don't transmit s. This (potentially) breaks the subgraph into different components. This algorithm is basically a variant of the *block tree decomposition of a graph* (Acemoglu, Malekian, and Ozdaglar (2016)). The following example provides a simple illustration of how the algorithm works.

Example 5. *Suppose $N = \{1,2,3\}$ with 1 and 3 being biased, and 2 being unbiased. Consider the line graph $g = \{12, 23\}$. The original graph G is given by:*

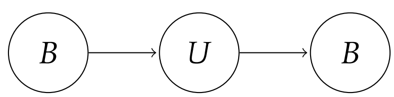

FIGURE 14: Original grpah G

Assume that $\pi(1|0) > \bar{\pi}$ and the Designer targets 1. Then the algorithm returns $G^* = \{12\}$. The graph returned by the algorithm G^* when the Designer targets 1 is given in figure **??**.

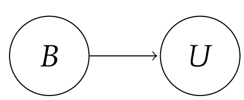

FIGURE 15: G^* when $t = \{1\}$.

If the Designer targets 2, the algorithm returns $G^* = \emptyset$.

I first define the notion of diffusion centrality and then discuss how it relates to the influence measure is this paper. The following definition has been adopted from Banerjee et al. (2014)

Definition 6. *Consider a network $G = \{0,1\}^{N\times N}$ and suppose each agent transmits information with probability p independent across history and neighbors. The Diffusion Centrality of agent i is the expected total number of times that a piece of information originating from i is heard by any agent in T communication rounds*

$$DC_G^T = \left(\sum_{t=1}^{T}(pG)^t\right).\mathbf{1}$$

Notice the similarity with the definition of influence. The "piece of information" in our context is replaced by "agent taking high action". The primary difference, however, is that the probability of information transmission (p) is endogenous in our model. This can cause the two measures to differ significantly. The following result shows when the notion of influence and diffusion centrality coincide.

Lemma 8. *Suppose $\sigma_{ij}^{\tau} = \sigma$ for all $i,j \in N$, $\tau = 1,\ldots,n-1$ and all information structures $\{S,\pi\}$. Then*

$$C_i(G,\{S,\pi\}) = \lambda[DC_G^{n-1}]_i$$

where $\lambda > 0$ is a constant scaling factor.

Proof. Suppose agent i is seeded with a signal realization s. Consider an agent j connected to i with a directed path w of length l_w. By the above condition, the signal reaches j with probability σ^{l_w}. Then the influence of agent i is given by

$$\begin{aligned}
C_i(G,\{S,\pi\}) &= \sum_{j\in N}\sum_{s} Pr[s]\left[\sum_{w\in W_{ij}}\prod_{k=1}^{l_w-1}\sigma_{i_k i_{k+1}}^{\tau}\mathbf{1}_{a\geq\kappa}\right] \\
&= \sum_{j\in N}\sum_{s} Pr[s]\left[\sum_{w\in W_{ij}}\sigma^{l_w}\mathbf{1}_{a\geq\kappa}\right] \\
&= \sum_{j\in N}\sum_{s:a(s)\geq\kappa} Pr[s]\left[\sum_{w\in W_{ij}}\sigma^{l_w}\right]
\end{aligned}$$

$$= \left\{ \sum_{s:a(s)\geq\kappa} Pr[s] \right\} \underbrace{\left[\sum_{j\in N} \sum_{w\in W_{ij}} \sigma^{l_w} \right]}_{(A)}$$

where l_w is the length of the path w from i to j. The only thing left to show is that the expression (A) corresponds to the diffusion centrality of agent i. To this end, note that any graph $G \subset N^2$ can be equivalently represented by the adjacency matrix $H = \{0,1\}^{N\times N}$ where $(i,j) \in G$ if and only if $H_{ij} = 1$. Then

$$\begin{aligned} \sum_{j\in N} \sum_{w\in W_{ij}} \sigma^{l_w} &= \sum_{j\in N} \left[\sum_t (\sigma H)^t \right]_{ij} \\ &= \left[\left(\sum_{t=1}^{n-1} (\sigma H)^t \right) . \mathbf{1} \right]_i \\ &= [DC_G^{n-1}]_i \end{aligned}$$

Combining the above two expressions we have

$$C_i(G, \{S,\pi\}) = \lambda \left[DC_G^{n-1} \right]_i$$

where $\lambda = \sum_{s:a(s)\geq\kappa} Pr[s]$ is the total probability of signals that induce the agents to choose $a \geq \kappa$. □

7.2 Proofs of Section 4

Proof of lemma 1. Fix an information structure (S_t, π) and suppose the seed t observes the signal $s \in S_t$. Take any receiver $i \in G_t$, i.e. there exists a path $P = (i_0, i_1 \ldots, i_K)$ of length $K+1$ with $i_0 = t$ and $i_K = i$. Since G is acyclic, this path is unique. Consequently any signal that i observes must be transmitted by receivers along this path P. Conditional on receiving a message s' from receiver i_{k-1} in round $k-1$, let $\sigma_{i_k i_{k+1}}(s|s')$ be the probability that receiver i_k send a message $m_{i_k i_{k+1}} = s$ to receiver i_{k+1}. Then the posterior belief receiver i forms upon receiving signal s is given by Bayes' rule as follows

$$\mu^i(\omega = 1|s) = \frac{\prod_{k=0}^{K-1} \overbrace{\left[\sum_{s' \in S_t} \sigma_{i_k i_{k+1}}(s|s')\pi(s'|1)\right]}^{Pr[i_k \text{ sends } s \text{ to } i_{k+1} \text{ given } \omega = 1]} \mu_0(\omega = 1)}{\underbrace{\sum_{\omega}\left\{\prod_{k=0}^{K-1}\left[\sum_{s'} \sigma_{i_k i_{k+1}}(s|s')\pi(s'|\omega)\right]\right\}\mu_0(\omega)}_{Pr[i \text{ receives } s]}}$$

as long as i receives s with positive probability. Thus following any history h_i^{n-1} where receiver i observes signal $s \in S_t$, his expected utility is given by

$$\begin{aligned} U_i(\mathbf{a}|h_i^{n-1}) &= \mathbb{E}_{\mu^i}\left[-\sum_{j \in i \cup G_i^{out}} (a_j - \omega - b_i)^2\right] \\ &= -\sum_{j \in i \cup G_i^{out}} \left[\mu^i(\omega = 1|s)(a_j - 1 - b_i)^2 + (1 - \mu^i(\omega = 1|s))(a_j - b_i)^2\right] \end{aligned}$$

The objective of receiver i following the communication round is then as follows

$$\max_{a_i \in A} U_i(\mathbf{a}|h_i^{n-1})$$

Note that the constraint set A is compact and $U_i(\mathbf{a}|h_i^{n-1})$ is continuous and strictly concave in a_i. By Weierstrass theorem, this is a well-defined optimisation problem whose unique solution is obtained from the first order necessary condition

$$\begin{aligned} 2\mu^i(\omega = 1|s)(a_i - 1 - b_i) &= -2(1 - \mu^i(\omega = 1|s))(a_i - b_i) \\ \therefore a_i^*(h_i^{n-1}) &= \mu^i(\omega = 1|s) + b_i. \end{aligned}$$

□

Proof of proposition 1. Fix an information structure (S_t, π). Consider a receiver $i \in G_t$ and a communication round τ such that $h_i^{\tau-1} \backslash h_i^{\tau-2} = s \in S_t$. receiver i is an *active* receiver in round τ and can thus send a message $m_{ij} \in \bar{S}_t$ to each of his neighbour $j \in G_i^{out}$.

We know that the optimal action for receiver i upon observing a history $h_i^{n-1} = \{s\}$ is

$a_i^*(h_i^{n-1}) = \mu^i(\omega = 1|s) + b_i$. The same logic as in Lemma 1 implies that j can only receive information from receiver i (and no one else).[23]

The objective is to characterise the equilibrium truthful network arising from cheap-talk communication. Suppose j believes that i communicates all signals truthfully to him. (i.e. $i, j \in G^{truth}$). Then the posterior formed by receiver j upon observing message $m_{ij} = s'$ from i is $\mu^j(\omega = 1|s')$. The expected payoff to receiver i from transmitting signal s' is

$$
\begin{aligned}
& -\mu^i(\omega = 1|s)\Big\{(a_i^* - 1 - b_i)^2 + (a_j^* - 1 - b_i)^2 + \sum_{k \in G_i^{out} \setminus \{i,j\}} (a_k^* - 1 - b_i)^2\Big\} - \\
& \mu^i(\omega = 0|s)\Big\{(a_i^* - b_i)^2 + (a_j^* - b_i)^2 + \sum_{k \in G_i^{out} \setminus \{i,j\}} (a_k^* - b_i)^2\Big\} \\
= & -\mu^i(\omega = 1|s)\Big\{(\mu^i(\omega = 1|s) - 1)^2 + (\mu^j(\omega = 1|s') + b_j - 1 - b_i)^2 \\
& + \sum_{k \in G_i^{out} \setminus \{i,j\}} (a_k^* - 1 - b_i)^2\Big\} - [1 - \mu^i(\omega = 1|s)]\Big\{(\mu^i(\omega = 1|s))^2 \\
& + (\mu^j(\omega = 1|s') + b_j - b_i)^2 + \sum_{k \in G_i^{out} \setminus \{i,j\}} (a_k^* - b_i)^2\Big\}
\end{aligned}
$$

Hence receiver i is willing to truthfully communicate signal s to receiver j if and only if he does not profitably deviate by sending any other signal $\hat{s} \in \bar{S}_t$. That is,

$$
\begin{aligned}
& -\mu^i(\omega = 1|s)\Big\{(\mu^i(\omega = 1|s) - 1)^2 + (\mu^j(\omega = 1|s) + b_j - 1 - b_i)^2 \\
& + \sum_{k \in G_i^{out} \setminus \{i,j\}} (a_k^* - 1 - b_i)^2\Big\} - [1 - \mu^i(\omega = 1|s)]\Big\{(\mu^i(\omega = 1|s))^2 \\
& + (\mu^j(\omega = 1|s) + b_j - b_i)^2 + \sum_{k \in G_i^{out} \setminus \{i,j\}} (a_k^* - b_i)^2\Big\} \\
\geq \quad & -\mu^i(\omega = 1|s)\Big\{(\mu^i(\omega = 1|s) - 1)^2 + (\mu^j(\omega = 1|s') + b_j - 1 - b_i)^2
\end{aligned}
$$

[23]Since i observes a non-null history in round $\tau - 1$, it must be that i lies along the *unique* path from the target t and receiver j.

$$
\begin{aligned}
&+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-1-b_i)^2\Bigg\}-[1-\mu^i(\omega=1|s)]\Bigg\{(\mu^i(\omega=1|s))^2\\
&+(\mu^j(\omega=1|s')+b_j-b_i)^2+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-b_i)^2\Bigg\}\\
\Longrightarrow\ &\mu^i(\omega=1|s)\Bigg\{(\mu^j(\omega=1|s')+b_j-1-b_i)^2-(\mu^j(\omega=1|s)+b_j-1-b_i)^2\Bigg\}\\
&+[1-\mu^i(\omega=1|s)]\Bigg\{(\mu^j(\omega=1|s')+b_j-b_i)^2-(\mu^j(\omega=1|s)+b_j-b_i)^2\Bigg\}\\
&\geq 0\\
\Longrightarrow\ &\mu^i(\omega=1|s)\Bigg\{[\mu^j(\omega=1|s')-\mu^j(\omega=1|s)][\mu^j(\omega=1|s')+\mu^j(\omega=1|s)\\
&+2(b_j-1-b_i)]\Bigg\}+[1-\mu^i(\omega=1|s)]\Bigg\{[\mu^j(\omega=1|s')\\
&-\mu^j(\omega=1|s)][\mu^j(\omega=1|s')+\mu^j(\omega=1|s)+2(b_j-b_i)]\Bigg\}\\
&\geq 0\\
\Longrightarrow\ &[\mu^i(\omega=1|s')-\mu^i(\omega=1|s)]\{\mu^i(\omega=1|s)[\mu^j(\omega=1|s')+\mu^j(\omega=1|s)\\
&+2(b_j-1-b_i)]+(1-\mu^i(\omega=1|s))[\mu^j(\omega=1|s')+\mu^j(\omega=1|s)+2(b_j-b_i)]\}\\
&\geq 0\\
\Longrightarrow\ &[\mu^i(\omega=1|s')-\mu^i(\omega=1|s)]\{\mu^i(\omega=1|s')+\mu^i(\omega=1|s)\\
&+2(b_j-b_i)-\mu^i(\omega=1|s)\}\\
&\geq 0\\
\Longrightarrow\ &[\mu^i(\omega=1|s')-\mu^i(\omega=1|s)]\{\mu^i(\omega=1|s')-\mu^i(\omega=1|s)+2(b_j-b_i)\}\geq 0
\end{aligned}
$$

where the second last line makes use of the fact that $\mu^j(\omega=1|s)=\mu^i(\omega=1|s)$ under truthful communication. Thus the condition for truthful communication of signal s by receiver i to j is

$$
\begin{cases}
2(b_j-b_i)>-[\mu^i(\omega=1|s')-\mu^i(\omega=1|s)] & \text{if } \mu^i(\omega=1|s')>\mu^i(\omega=1|s)\\
2(b_j-b_i)<[\mu^i(\omega=1|s)-\mu^i(\omega=1|s')] & \text{if } \mu^i(\omega=1|s')<\mu^i(\omega=1|s)
\end{cases}
\tag{12}
$$

Replacing s and s' above yields the condition for the truthful communication of s'. Thus for any pair of signals $s,s'\in\bar{S}_t$ with $\mu^i(\omega=1|s)>\mu^i(\omega=1|s')$, receiver i truthfully communicates both the signals to j if and only if

$$2|(b_j - b_i)| \leq \mu^i(\omega = 1|s) - \mu^i(\omega = 1|s')$$

Iterating over all such pairs of signals characterises the condition for truthful communication of all signals by receiver i to his neighbour j

$$2|(b_j - b_i)| \leq min\{\mu^i(\omega = 1|s) \geq \mu^i(\omega = 1|s') : s, s' \in \bar{S}_t\} \tag{13}$$

□

Proof of lemma 2. Fix an information structure (S_t, π) with $S_t = \{l, h\}$. Consider a receiver $i \in G_i^{truth}$, i.e. i truthfully receives any signal that the seed t observes. By the Bayes plausibility condition, we must have $\mu^i(\omega = 1|l) \leq \mu_0(\omega = 1) \leq \mu^i(\omega = 1|h)$. Consequently any receiver who observes the l signal takes the "low" action: $a_i^* = \mu(\omega = 1|l) + b_i < \mu_0(\omega = 1) + b_i < \kappa$.

Now consider the case where receiver i observes the signal h. Under truthful communication, the posterior formed by receiver i is thus

$$\mu^i(\omega = 1|h) = \frac{\mu_0(\omega = 1)\pi(h|1)}{\mu_0(\omega = 1)\pi(h|1) + \mu_0(\omega = 0)\pi(h|0)}$$

and the optimal action is $a_i^*(h) = \mu^i(\omega = 1|h) + b_i$. Thus receiver i chooses the "high action" upon observing h if and only if

$$\begin{aligned}
&\kappa \leq a_i^*(h) \\
\implies &\kappa \leq \frac{\mu_0(\omega = 1)\pi(h|1)}{\mu_0(\omega = 1)\pi(h|1) + \mu_0(\omega = 0)\pi(h|0)} + b_i \\
\implies &\kappa \leq \frac{1}{1 + \frac{\mu_0(0)}{\mu_0(1)}\frac{\pi(h|\omega=0)}{\pi(h|1)}} + b_i \\
\implies &\frac{\pi(h|0)}{\pi(h|1)} \leq \frac{\mu_0(\omega = 1)}{\mu_0(\omega = 0)}\left[\frac{1 - \kappa + b_i}{\kappa - b_i}\right]
\end{aligned}$$

Replacing the vale of b_i for the unbiased and biased receivers yield the cutoffs π^* and

π^{**}. The difference in the threshold is given by

$$\pi^{**} - \pi^{*} = \frac{\mu_0(\omega = 1)}{\mu_0(\omega = 0)}\left[\frac{1}{\kappa - b} - \frac{1}{\kappa}\right]$$

The expression is increasing in b, $\frac{\partial(\pi^{**}-\pi^{*})}{\partial b} = \frac{\mu_0(\omega=1)}{\mu_0(\omega=0)}\frac{1}{(\kappa-b)^2} > 0$. With $S_t = \{l, h\}$, the condition for truthful communication from proposition **??** simplifies to

$$2|(b_j - b_i)| \leq \mu^i(\omega = 1|h) - \mu^i(\omega = 1|l)$$

Using the expressions for the posterior beliefs under truthful communication, we get the cutoff

$$2|(b_j - b_i)| \leq \frac{\mu_0(\omega = 1)\pi(h|1)}{\mu_0(\omega = 1)\pi(h|1) + \mu_0(\omega = 0)\pi(h|0)} - \frac{\mu_0(\omega = 1)\pi(l|1)}{\mu_0(\omega = 1)\pi(l|1) + \mu_0(\omega = 0)\pi(l|0)}$$

$$2|(b_j - b_i)| \leq \frac{\frac{\mu_0(\omega=0)}{\mu_0(\omega=1)}\left[\frac{\pi(l|0)}{\pi(l|1)} - \frac{\pi(h|0)}{\pi(h|1)}\right]}{\left(1 + \frac{\mu_0(\omega=0)}{\mu_0(\omega=1)}\frac{\pi(h|0)}{\pi(h|1)}\right)\left(1 + \frac{\mu_0(\omega=0)}{\mu_0(\omega=1)}\frac{\pi(l|0)}{\pi(l|1)}\right)}$$

□

Proof of lemma 3. Fix an information structure (S_t, π) with $S_t = \{l, h\}$ and $\pi(h|1) = 1$. Under such an information structure, realisation of l signal reveals the state: $\mu^i(\omega = 1|l) = 0$. Hence the condition for truthful communication between i and j simplifies to

$$2|(b_j - b_i)| \leq \frac{1}{1 + \frac{\mu_0(\omega=0)}{\mu_0(\omega=1)}\frac{\pi(h|0)}{\pi(h|1)}}$$

$$\implies 1 + \frac{\mu_0(\omega = 0)}{\mu_0(\omega = 1)}\frac{\pi(h|0)}{\pi(h|1)} \leq \frac{1}{2|(b_j - b_i)|}$$

$$\implies \frac{\pi(h|0)}{\pi(h|1)} \leq \frac{\mu_0(\omega = 1)}{\mu(\omega = 0)}\left[\frac{1}{2|(b_j - b_i)|} - 1\right]$$

Notice that $|(b_j - b_i)| \in \{0, b\}$. When $|(b_j - b_i)| = 0$, i.e. i and j have the same bias, the conditions for truthful communication are trivially satisfied. On the other hand, the condition for truthful communication between receivers of different bias is given by

$$\frac{\pi(h|0)}{\pi(h|1)} \leq \frac{\mu_0(\omega=1)}{\mu(\omega=0)}\left[\frac{1}{2b}-1\right] \equiv \bar{\pi}$$

Under Assumption 3, we know that $\pi^* < \pi^{**}$:

$$\begin{aligned} & b > \frac{\kappa}{2} \\ \implies & 2b > \kappa \\ \implies & \left[\frac{1}{2b}-1\right] > \left[\frac{1}{\kappa}-1\right] \\ \implies & \pi^* > \bar{\pi} \end{aligned}$$

This yields the following conditions

$$\bar{\pi} < \pi^* < \pi^{**}$$

The corollary then follows directly from proposition Corollary 1, and lemma 2. Suppose the precision of h signal is such that $\frac{\pi(h|\omega=0)}{\pi(h|\omega=1)} < \bar{\pi}$. In this case $a_j(h) \geq \kappa$ for all j. Moreover, the signal is precise enough for every receiver to truthfully communicate with each other, $\sigma_{ij}(s|s) = 1$ for all $i, \in G_t$ and $j \in G_i^{out}$. Now let $\pi^* < \frac{\pi(h|\omega=0)}{\pi(h|\omega=1)} < \pi^{**}$.

In this case all receivers with the same bias communicate truthfully with each other. However, only biased receivers choose the high action at this precision level. Analogous arguments for the rest of the cases yield the conditions in the corollary. □

Proof of proposition 2. The objective is to construct the sender's payoff and optimal information for each possible seed $t \in N$. I prove this in two ways - an algebraic method, and a geometric method. The sender's payoff is given by

$$\bar{v}(p) = \begin{cases} 0 & \text{if } p \in [0, \mu(t)) \\ v(G_t^{truth}) & \text{if } p \in [\mu(t), 2b) \\ v(G_t) & \text{if } p \in [2b, 1] \end{cases}$$

We can rewrite the sender's payoff conditional on signals realised. All receivers take the "low" action following a l signal, yielding a payoff of $V(l) = 0$. Similarly, the payoff conditional on h signal is

$$V(h) = \begin{cases} 0 & \text{if } \pi(h|0) \in [\bar{\pi}, 1] \\ v(G_t^{truth}) & \text{if } \pi(h|0) \in [\pi(t), \bar{\pi}) \\ v(G_t) & \text{if } \pi(h|0) \in [0, \pi(t)) \end{cases}$$

where $\pi(t) \in \{\pi^*, \pi^{**}\}$. Thus the ex-ante payoff to the sender from an information structure (S_t, π) is given by

$$[\mu_0(\omega = 1) + \mu_0(\omega = 0)\pi(h|0)]V(h) + [\mu_0(\omega = 0)(1 - \pi(h|0))]V(l)$$

and the optimal information solves the following optimization problem

$$\max_{\pi(h|0) \in [0,1]} [\mu_0(\omega = 1) + \mu_0(\omega = 0)\pi(h|0)]V(h)$$

STEP 1: *The optimisation problem is well defined.*
The constraint set $[0,1]$ is compact. In order to show that a maxima exists in this set, I need show that the objective function is *upper semi-continuous*. The upper semi-continuity follows from the standard tie-breaking assumption in the bayesian persuasion literature. The following figure illustrates an example of the objective function of the above optimisation problem.

STEP 2: *The optimal precision must belong to the set $\pi^*(h|0) \in \{\pi(t), \bar{\pi}\}$*
Suppose $\pi^*(h|0) \notin \{\pi(t), \bar{\pi}\}$. Let the optimal solution $\pi(h|0)^* \in [0, \pi(t))$. The expected payoff to the sender is thus $[\mu_0(\omega = 1) + \mu_0(\omega = 0)\pi(h|0)^*]v(G_t)$. Consider a deviation $\pi(\tilde{h}|0) = \pi(h|0)^* + \epsilon$ for $\epsilon > 0$ small enough. Such a deviation does not impact the total number of receivers taking the high action, but increases the probability of h signal being realized:

$$\begin{aligned}[\mu_0(\omega = 1) + \mu_0(\omega = 0)\pi(\tilde{h}|0)]v(G_t) &= [\mu_0(\omega = 1) + \mu_0(\omega = 0)(\pi(h|0)^* + \epsilon)]v(G_t) \\ &> [\mu_0(\omega = 1) + \mu_0(\omega = 0)\pi(h|0)^*]v(G_t)\end{aligned}$$

Hence $\pi(h|0)^*$ is not optimal, a contradiction. The other cases can be solved analogously with the same logic: we can increase the probability of h signal without changing the receivers' optimal strategies.

STEP 3: *Find the optimality conditions*

From Step 2, all we need to check is which of the values $\{\pi(t), \bar{\pi}\}$ gives the sender the highest utility. At $\bar{\pi}$, the expected utility is $[\mu_0(\omega = 1) + \mu_0(\omega = 0)\bar{\pi}]v(G_t) = \frac{\mu_0(\omega=1)}{2b}v(G_t)$.

On the other hand, the expected utility at $\pi^*(h|0) = \pi(t)$ is $\frac{\mu_0(\omega=1)}{\mu(t)}v(G_t^{truth})$. The optimal information structure is defined by $\beta = max\left\{\frac{v(G_t^{truth})}{\mu(t)}, \frac{v(G_t)}{2b}\right\}$, giving the sender a payoff of $F_{(S_t,\pi)(i)}(\mu) = \beta\mu_0(\omega = 1)$. The corresponding optimal information structure can be found by Bayes' rule.

- Suppose $\frac{v(G_t^{truth})}{\mu(t)} > \frac{v(G_t)}{2b}$. Then the optimal information structure involves $\pi^*(h|0) = \pi(t)$. The l signal induces a posterior belief $\mu(\omega = 1|l) = 0$ while the h signal induces a posterior belief $\mu(\omega = 1|h) = \mu(t)$.
- Suppose $\frac{v(G_t^{truth})}{\mu(t)} < \frac{v(G_t)}{2b}$. The optimal information structure involves $\pi^*(h|0) = \bar{\pi}$. The l signal induces a posterior belief $\mu(\omega = 1|l) = 0$ while the h signal induces a posterior belief $\mu(\omega = 1|h) = 2b$.
- When $\frac{v(G_t^{truth})}{\mu(t)} = \frac{v(G_t)}{2b}$, the sender is indifferent between $\bar{\pi}$ and $\pi(t)$. Any linear combination of these two information structures is optimal.

The geometric proof involves the standard concavification approach. The key idea is that β parameter defined above corresponds to the slope of the concave closure of the sender's payoff. □

Proof of proposition 3. Lemma 3 gives the expression for the sender's payoff when he targets receiver t: $F_{(S_t,\pi)} = \beta_t\mu_0(\omega = 1)$. Let (S_t^*, π^*) denote the corresponding optimal information structure and $\bar{\chi}_1 = \{(S_t^*, \pi^*)\}_{t\in N}$ be the set of optimal information provided for each possible choice of target. The targeting problem for the sender is Then

$$\begin{aligned}\max_{\psi\in\Delta(\bar{\chi}_1)} &\sum_{(S_t^*,\pi^*)\in\bar{\chi}_1} \psi\Big((S_t^*, \pi^*)\Big)F_{(S_t^*,\pi^*)}(\mu_0) \\ \equiv&\mu_0(\omega = 1)\sum_{(S_t^*,\pi^*)\in\bar{\chi}_1} \psi\Big((S_t^*, \pi^*)\Big)\beta_t\end{aligned}$$

The objective function is linear in ψ. Hence the optimal solution involves putting all weight on the coefficient with the highest value of β. This set is characterised as follows:

$$\chi = \left\{ (S^*_{t^*}, \pi^*) : t^* \in \max_{t \in N} \left\{ \max \left\{ \frac{v(G_t^{truth})}{\mu(t)}, \frac{v(G_t)}{2b} \right\} \right\} \right\}$$

This is the set of information structures that yield the highest payoff to the sender. Hence any optimal solution ψ^* must put all probability mass on this set. This yields the following solution to the targeting problem

$$\psi^*\Big((S^*_t, \pi^*)\Big) = \begin{cases} \frac{1}{|\chi|} & \text{if } (S^*_t, \pi^*) \in \chi \\ 0 & \text{o.w.} \end{cases}$$

The converse is straightforward. Redistributing the weights among the information structures in χ^*_T does not change the payoff to the sender (otherwise the information structure from which the sender deviates should not belong to χ). Moreover, shifting $\epsilon > 0$ weight to some $(S^*_t, \pi^*) \notin \chi$ strictly reduces the sender's payoff

$$(1-\epsilon)F_{(S^*_{t^*}, \pi)}(\mu_0) + \epsilon F_{(S^*_t, \pi^*)}(\mu_0) < F_{(S^*_{t^*}, \pi^*)}(\mu_0)$$

since $F_{(S^*_t, \pi^*)}(G) < F_{(S^*_{t^*}, \pi^*)}(\mu_0)$ by definition. □

Proof of Theorem 1. I first give a more explicit characterisation of influence of a receiver in terms of the network structure and the strategies of the players. Then I show that the set of receivers with the highest influence coincides with the set of receivers that the sender obtains the highest payoff from (χ^*_T). The definition of influence of a receiver reduces to the following

$$\begin{aligned} C_t(G, (S_t, \pi)) &= \sum_{j \in N} \left[Pr\{h\} \left\{ \prod_{k=0}^{l_j - 1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_j \geq \kappa} \right\} + Pr\{l\} \left\{ \prod_{k=0}^{l_j - 1} \sigma_{i_k i_{k+1}}(l|l) \mathbf{1}_{a_j \geq \kappa} \right\} \right] \\ &= \sum_{j \in N} \left[Pr\{h\} \left\{ \prod_{k=0}^{l_j - 1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_j \geq \kappa} \right\} \right] \\ &= \sum_{j \in G_t^{truth}} \left[Pr\{h\} \left\{ \prod_{k=0}^{l_j - 1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_j \geq \kappa} \right\} \right] \end{aligned}$$

$$
+ \underbrace{\sum_{j\in N\setminus G_t^{truth}} \left[Pr\{h\} \left\{ \prod_{k=0}^{l_j-1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_j \geq \kappa} \right\} \right]}_{=0}
$$

$$
= \sum_{j\in G_t^{truth}} \left[Pr\{h\} \left\{ \prod_{k=0}^{l_j-1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_j \geq \kappa} \right\} \right]
$$

$$
= \left[Pr\{h\} \sum_{j\in G_t^{truth}} \left\{ \prod_{k=0}^{l_j-1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_j \geq \kappa} \right\} \right]
$$

$$
= \left[Pr\{h\} \left(\sum_{j\in G_t^{truth}\cap B} \left\{ \prod_{k=0}^{l_j-1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_j \geq \kappa} \right\} \right.\right.
$$

$$
\left.\left. + \sum_{r\in G_t^{truth}\cap U} \left\{ \prod_{k=0}^{l_r-1} \sigma_{i_k i_{k+1}}(h|h) \mathbf{1}_{a_r \geq \kappa} \right\} \right) \right]
$$

$$
= Pr\{h\} \left[\sum_{j\in G_i^{truth}\cap B} \mathbf{1}_{a_j \geq \kappa} + \sum_{j\in G_i^{truth}\cap U} \mathbf{1}_{a_j \geq \kappa} \right]
$$

$$
= Pr\{h\} \left[|G_t^{truth} \cap B| + |G_t^{truth} \cap U| \right]
$$

Define the set $\bar{X} = \left\{ C_t(G, (S_t^*, \pi^*)) \right\}_{t\in N}$ where (S_t^*, π^*) is the optimal information structure obtained from Prooposition 2. Notice that $|\bar{X}| = n$. Let $X^* = \max \bar{X}$. To complete the proof, I just need to show that $X^* = \chi$.

The key to the set equivalence is to construct the one-one mapping between the measure of influence and the sender's payoff. Let A_t be the set of receivers who choose the high action when the sender chooses (S_t^*, π^*). This is exactly what $V(h)$ measures. We thus have the following:

$$
\begin{aligned}
C_t(G, (S_t^*, \pi^*)) &= Pr(h)|A_t| \\
&= [\mu_0(\omega = 1) + \mu_0(\omega = 1)\pi^*(h|0)]V(h) \\
&= F_{(S_t^*, \pi^*)}(\mu_0)
\end{aligned}
$$

Take any $C_t(G,(S_t^*,\pi^*)) \in X^*$. By definition we have

$$C_t(G,(S_t^*,\pi^*)) \geq C_j(G,(S_j^*,\pi^*)) \;\; \forall\, j \in N$$
$$\leftrightarrow F_{(S_t^*,\pi^*)}(\mu_0) \geq F_{(S_t^*,\pi^*)}(\mu_0) \;\; \forall\, j \in N$$

Hence $(S_t^*,\pi^*) \in \chi$, and $\psi^*\Bigg((S_t^*,\pi^*)\Bigg) > 0$. The converse follows similarly. As a result,

$$\psi^*\Bigg((S_t^*,\pi^*)\Bigg) > 0 \leftrightarrow C_t(G,(S_t^*,\pi^*)) \in X^*$$

This concludes the proof. □

Proof of Lemma 4. Suppose $b_i = b_j = 0$ for all $i,j \in N$, and G is strongly connected. Since information is verifiable, Proposition 4 tells us that all receivers in the network receive the information that the seed t observes. Hence all receivers either take the high action, or they all take the low action. Thus the sender's payoff is given by:

$$\tilde{v}(p) = \begin{cases} 0 & \text{if } p \in [0,\kappa) \\ v(N) & \text{if } p \in [\kappa,1] \end{cases}$$

Thus the sender's payoff following any target $t \in N$ is given by

$$F_{(S_t^*,\pi^*)}(\mu_0) = \mu_0(\omega = 1)\frac{v(N)}{\kappa}$$

Hence he is indifferent between any choice of target. The optimal information structure involves $\pi^*(h|0) = \pi^*$. □

7.3 Proofs of Section 5

Proof of theorem 2. Let G and G' be two networks of n receivers such that $b_i = b_j$ for all $i,j \in N$. Suppose G' is more reachable than G, that is, $G_i \subseteq G_i'$ for all $i \in N$ and $G_j \subset G_j'$ for at least one receiver j. Notice that the optimal information structure in this setting is given by which is independent of the network structure. This gives the sender an expected payoff of $\bar{v}(G) = \Bigg[\mu(\omega = 1) + \mu(\omega = 0)\pi^*\Bigg]v(G_{t^*})$ where $DC_G(t^*) \geq DC_G(t)$

		S^*	
		0	1
ω	0	$1-\pi^*$	π^*
	1	0	1

for all $t, t^* \in T$. Now consider the sender's problem in the network G'. The optimal information structure remains the same. Before proceeding, it is clear that the following result holds

$$G' \text{ is more reachable than } G \leftrightarrow [DC_{G'}(i)]_{i\in N} \geq [DC_G(i)]_{i\in N}$$

The result then follows from lemma 6. Since $[DC_{G'}(i)]_{i\in N} \geq [DC_G(i)]_{i\in N}$, it must be that $\max\{DC_{G'}(i)\}_{i\in N} \geq \max\{DC_G(i)\}_{i\in N}$. This implies that $v(G_{t'}) \geq v(G_t)$. I can therefore conclude that $\bar{v}(G') \geq \bar{v}(G)$. □

7.4 Alternate protocol: Verifiable communication case

Suppose that receivers cannot falsify the information content during the communication phase, in contrast to the cheap talk scenario analysed previously.[24] However receivers can block signals from being transmitted, i.e. $M_{ij}(s) \equiv \{s, \emptyset\}$ whenever i receives signal s. What does this restriction imply for the analysis? Recall the payoff function of receiver i: $-(a_1-\omega-b_i)^2 - \sum_{j\in G_i^{out}}(a_j-\omega-b_i)^2$. When he is communicating with his neighbour j, the relevant aspect that determines the strategy is $-(a_j-\omega-b_i)^2$. Hence i can influence neighbour j's action only through communication. Consequently i and j engage in a disclosure game analysed in Milgrom (1981)) under the verifiable communication protocol. I refer to receiver holding information in a communication round as the *sender*, while the neighbour he communicates to as the *receiver*.

7.5 Communication stage

The communication phase is triggered once the network is seeded with a signal $s \in S_t$. Since the sender can seed only one receiver, there is exactly one signal flowing through the network. The communication strategy then reduces to the following question:

[24] As before, I assume that the identity source cannot be falsified. In our media outlet parable, an example of verifiable communication would be "retweeting"/sharing the link to the web article.

conditional on receiving signal s, what message should he send to each of his neighbours? This is answered in the following proposition, which states that precision of information does not affect information diffusion under verifiable communication

Proposition 4. *Fix an information structure (S_t, π). Consider a receiver i and a communication round τ such that $h_i^{\tau-1} \backslash h_i^{\tau-2} = (s)$. The optimal communication strategy for receiver i is then given as follows:*

$$\sigma_{ij}(h_i^\tau) = s \;\forall\; s \in S_t,\; \forall\; j \in G_i^{out} \tag{14}$$

Each receiver mechanically transmits whatever information he receives to all his neighbours. This simplifies the sender's problem since he does not have to incorporate incentives to transmit information. The logic is exactly the same as the *unraveling argument* as in Milgrom (1981). Senders and receivers with the same bias communicate truthfully since their preferences are perfectly aligned. Now consider the case where the sender is unbiased while the receiver is biased. The idea is as follows. Whenever the unbiased receiver blocks a signal, the receiver forms *skeptical beliefs*. That is, the receiver believes that the blocked signal is always the one that induces the highest posterior $\mu(\omega = 1|\bar{s}) \equiv \max_{s \in S_t}\{\mu(\omega = 1|s)\}$. In other words, the biased receiver takes the highest possible action if the unbiased sender blocks information to him. Revealing signal s leads to a lower action from the receiver, which is closer to the sender's bliss point. Hence the sender strictly prefers to reveal s. Analogous arguments can be made for a biased sender and unbiased receiver. Since all information is revealed in equilibrium, skeptical beliefs is consistent with equilibrium behaviour. What does this imply for the sender? When the sender seeds receiver t with signal s, every receiver in G_t receives the signal. All that matters to the sender is the number of biased and unbiased receivers in G_t.

7.6 Information design stage

The sender's analysis of the optimal information is simpler than that of the cheap talk case. The optimization problem for the sender is still given by 4. However, the sender's payoff is no longer endogenous to the information structure itself. This allows us to use standard concavification techniques developed in the persuasion literature. Since the sender doesn't need to incentivise the receivers to transmit information, the only thresholds that he needs to consider are π^* and π^{**}. This is summarised in the following corollary

Corollary 2. *Consider any information structure (S_t^*, π). All receivers choose the "Low" action upon receiving the signal s such that $\mu(\omega = 1|s) < \mu_0(\omega = 1)$. The optimal actions taken by the receiver following any signal s with $\mu(\omega = 1|s) > \mu_0(\omega = 1)$ is as follows:*

- *When $\frac{\pi(s|0)}{\pi(s|1)} \leq \pi^*$: all receivers choose the "High" action.*
- *When $\pi^* < \frac{\pi(s|0)}{\pi(s|1)} \leq \pi^{**}$: only biased receivers choose the "High" action.*
- *When $\frac{\pi(s|0)}{\pi(s|1)} \geq \pi^{**}$: no receivers choose the "High" action.*

Let us construct the sender's payoff $\bar{v}$. This maps the number of receivers choosing the "High" action for each possible posterior over the states. Let μ^* (μ^{**}) be the posterior induced by a signal s with $\frac{\pi(s|0)}{\pi(s|1)} = \pi^*$. When $p \in [0, \mu^*)$, no receiver takes the high action and thus $\bar{v} = 0$. When $p \in [\mu^*, \mu^{**})$, only the biased receivers in G_t takes the high action. This yields a payoff of $v(G_t \cap B)$. Lastly, the sender earns $v(G_t)$ when $p \in [\mu^{**}, 1]$. The $\bar{v}$ function can thus be summarised as follows

$$\bar{v}(p) = \begin{cases} 0 & \text{if } p \in [0, \mu^*) \\ v(G_t \cap B) & \text{if } p \in [\mu^*, \mu^{**}) \\ v(G_t) & \text{if } p \in [\mu^{**}, 1] \end{cases} \tag{15}$$

As discussed above, this $\bar{v}$ does not change with the information structure chosen by the sender, thereby allowing us to read off the optimal signals from its concave closure. Following Kamenica and Gentzkow (2011), I can restate the sender's optimization problem in terms of choice of posteriors induced over the states. Hence the sender chooses a distribution over posteriors $\nu \in \Delta([0,1])$ to maximise

$$\max_{\nu \in \Delta([0,1])} \mathbb{E}_\nu[\bar{v}(p)]$$
$$\text{S.t. } \mathbb{E}_\nu[p] = \mu_0(\omega = 1)$$

The optimal solution to the above problem can be found by constructing the concave closure of the sender's payoff. The piecewise linear nature of $\bar{v}$ leads to a simple geometric characterisation of the optimal information structure provided to a target t.

Lemma 9. *Suppose the sender targets receiver t and define* $\beta = \max\left\{\frac{v(G_t \cap B)}{\mu^*}, \frac{v(G_t)}{\mu^{**}}\right\}$. *Then the maximum payoff that the sender obtains from targeting t is given by*

$$F_{(S_t^*,\pi)}(\mu_0) = \beta\mu_0(\omega = 1) \tag{16}$$

The optimal information structure involves two signals $S_t^* = \{l, h\}$ *each inducing a different posterior belief on* $\omega = 1$ *(*μ^l *and* μ^h*):*

$$(\mu^l, \mu^h) = \begin{cases} (0, \mu^{**}) & if\ \frac{v(G_t \cap B)}{v(G_t)} > \frac{\mu^{**}}{\mu^*} \\ \alpha(0, \mu^{**}) + (1-\alpha)(0, \mu^*) & if\ \frac{v(G_t \cap B)}{v(G_t)} = \frac{\mu^{**}}{\mu^*} \\ (0, \mu^*) & if\ \frac{v(G_t \cap B)}{v(G_t)} < \frac{\mu^{**}}{\mu^*} \end{cases}$$

for any $\alpha \in [0, 1]$.

The optimal targeting strategy of the sender can be characterised in terms of the influence of a receiver. Using analogous arguments to the cheap talk case, the following theorem characterises the optimal targeting strategy

Theorem 3. *Let* $\bar{\chi} = \{(S_t^*, \pi^*)\}_{t \in N}$ *be the set of optimal information structures sent to each target t. The optimal targeting strategy* $\psi^* \in \Delta(\bar{\chi})$ *is characterised by the following equivalence condition:*

$$\psi^*\Big((S_t^*, \pi^*)\Big) > 0 \leftrightarrow C_t\Big(G, (S_t^*, \pi^*)\Big) \geq C_j\Big(G, (S_j^*, \pi^*)\Big)\ \forall\ t, j \in N \tag{17}$$

The only difference between theorems 1 and 3 is the fact that the optimal target in theorem **??** involves choosing the receiver with the highest diffusion centrality, while in the latter case the optimal target is one with the highest influence. As has already been noted these two concepts may differ in any network. The difficulty arises since under verifiable communication, all receivers who receive the truthful signal need not take the same action. For instance, it is possible that only biased receivers choose the high action in equilibrium but the unbiased choose the low action despite both receiving the *h* signal. Influence measure the expected number of receivers who choose the high action from a seed, and thus has a one-one mapping between influence and the sender's payoff. Since the influence C_i of a receiver potentially changes with the information structure, the characterisation of the optimal strategies in the proposition is a bit subtle:

the equivalence holds only at the optimal information structures.

There is an alternate interpretation for the influence measure. Notice that for each signal realisation from (S,π), the communication strategy determines the extent to which the signal spread through the network. Hence each s (which can be interpreted as a "piece of information") can be associated with a diffusion centrality. Influence then measures an expected diffusion centrality for each receiver. This is in general complicated since not all receivers take the same action for a given signal realisation.

7.6.1 Proofs of Section 7.4

Proof of Proposition 4. Fix an information structure (S_t,π). consider and receiver i and a communication round τ such that $s \in h_i^{\tau-1} \backslash h_i^{\tau-2}$. From Lemma 1, the optimal action taken by i is $\mu^i(\omega=1|s)+b_i$, since s must be the true message under verifiable communication. Consider i's communication strategy with his neighbour $j \in G_i^{out}$ with bias b_j. Suppose receiver j has *skeptical beliefs*: if i blocks a message to him, j believes that i possesses the most unfavourable signal. That is, j's posterior would be $\mu^j(\omega=1|\bar{s}) \equiv \max_{s' \in \bar{S}_t}\{\mu^j(\omega=1|s')\}$.

Case 1: $b_i < b_j$ Suppose receiver i transmit the the signal s. His expected payoff is then

$$
\begin{aligned}
&-\mu^i(\omega=1|s)\Bigg\{(\mu^i(\omega=1|s)-1)^2+(\mu^j(\omega=1|s)+b_j-1-b_i)^2 \\
&+\sum_{k\in G_i^{out}\backslash\{i,j\}}(a_k^*-1-b_i)^2\Bigg\}-[1-\mu^i(\omega=1|s)]\Bigg\{(\mu^i(\omega=1|s))^2+(\mu^j(\omega=1|s) \\
&+b_j-b_i)^2+\sum_{k\in G_i^{out}\backslash\{i,j\}}(a_k^*-b_i)^2\Bigg\}
\end{aligned}
$$

If i blocks the signal s to receiver j, the latter chooses the action $a_i^j=\mu^j(\omega=1|\bar{s})+b_j$. Thus i's expected payoff is

$$
-\mu^i(\omega=1|s)\Bigg\{(\mu^i(\omega=1|s)-1)^2+(\mu^j(\omega=1|\bar{s})+b_j-1-b_i)^2
$$

$$+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-1-b_i)^2\Bigg\}-[1-\mu^i(\omega=1|s)]\Bigg\{(\mu^i(\omega=1|s))^2+(\mu^j(\omega=1|\bar{s})$$

$$+b_j-b_i)^2+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-b_i)^2\Bigg\}$$

Hence i is willing to transmit signal s to j if and only if

$$-\mu^i(\omega=1|s)\Bigg\{(\mu^i(\omega=1|s)-1)^2+(\mu^j(\omega=1|s)+b_j-1-b_i)^2$$

$$+(\mu^j(\omega=1|s)+b_j-b_i)^2+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-b_i)^2\Bigg\}+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-1-b_i)^2\Bigg\}$$

$$-[1-\mu^i(\omega=1|s)]\Bigg\{(\mu^i(\omega=1|s))^2$$

$$\geq\quad -\mu^i(\omega=1|s)\Bigg\{(\mu^i(\omega=1|s)-1)^2+(\mu^j(\omega=1|\bar{s})+b_j-1-b_i)^2$$

$$+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-1-b_i)^2\Bigg\}-[1-\mu^i(\omega=1|s)]\Bigg\{(\mu^i(\omega=1|s))^2+(\mu^j(\omega=1|\bar{s})$$

$$+b_j-b_i)^2+\sum_{k\in G_i^{out}\setminus\{i,j\}}(a_k^*-b_i)^2\Bigg\}$$

$$\Longrightarrow \mu^i(\omega=1|s)\Bigg\{[\mu^j(\omega=1|\bar{s})-\mu^j(\omega=1|s)][\mu^j(\omega=1|\bar{s})+\mu^j(\omega=1|s)$$

$$+2(b_j-1-b_i)]\Bigg\}+[1-\mu^i(\omega=1|s)]\Bigg\{[\mu^j(\omega=1|\bar{s})-\mu^j(\omega=1|s)][\mu^j(\omega=1|\bar{s})$$

$$+\mu^j(\omega=1|s)+2(b_j-b_i)]\Bigg\}$$

$$\geq 0$$

$$\Longrightarrow [\mu^i(\omega=1|\bar{s})-\mu^i(\omega=1|s)]\{\mu^i(\omega=1|\bar{s})+\mu^i(\omega=1|s)$$

$$+2(b_j-b_i)-\mu^i(\omega=1|s)\}$$

$$\geq 0$$

$$\Longrightarrow [\mu^i(\omega=1|\bar{s})-\mu^i(\omega=1|s)]\{\mu^i(\omega=1|\bar{s})-\mu^i(\omega=1|s)+2(b_j-b_i)\}\geq 0$$

The above expression is positive for any signal, by the definition of $\mu^i(\omega = 1|\bar{s})$. Hence receiver i transmits all signals $s \in \bar{S}_t$ to receiver j.

Case 2: $b_i > b_j$ In this case, j would have the skeptical belief $mu^j(\omega = 1|\underline{s}) \equiv \min_{s' \in \bar{S}_t}\{\mu^j(\omega = 1|s')\}$. The exact same calculations as in Case 1 shows that i finds it optimal to reveal all information to receiver j.

Case 3: $b_i = b_j$ Since i and j's preference are perfectly aligned, i has no incentive to block any signal to receiver j. Hence there is full revelation of all signals in this case.

Note that I have used the assumption that when indifferent, receivers transmit the signals to his neighbour. Thus all signals are revealed in the equilibrium of the communication game. In other words, blocking a signal is an off equilibrium path and hence beliefs are not pinned down by Bayes' rule. Thus skeptical beliefs is consistent with equilibrium behaviour. The logic is exactly as in Milgrom (1981). □

Proof of Corollary 2. From Proposition 4, we know that all signals that the seed t observes are revealed to every receiver $j \in G_t$. Thus the optimal action taken by receiver i upon observing signal s is $a_i^*(s) = \mu^i(\omega = 1|s) + b_i$. Consider the two cases:

Case 1: $\mu^i(\omega = 1|s) < \mu_0(\omega = 1)$ In this case, no receiver takes the "high" action upon observing the signal s under Assumption 3. This is because the optimal action is $\mu^i(\omega = 1|s) + b_i < \mu_o(\omega = 1) + b_i < \kappa$.

Case 2: $\mu^i(\omega = 1|s) > \mu_0(\omega = 1)$ The optimal action that i takes is $\mu^i(\omega = 1|s) + b_i$. hence receiver i chooses the "high" action if and only if

$$
\begin{aligned}
&\kappa \leq a_i^*(s) \\
\implies &\kappa \leq \frac{\mu_0(\omega = 1)\pi(s|1)}{\mu_0(\omega = 1)\pi(s|1) + \mu_0(\omega = 0)\pi(s|0)} + b_i \\
\implies &\kappa \leq \frac{1}{1 + \frac{\mu_0(0)}{\mu_0(1)}\frac{\pi(s|0)}{\pi(s|1)}} + b_i \\
\implies &\frac{\pi(s|0)}{\pi(s|1)} \leq \frac{\mu_0(\omega = 1)}{\mu_0(\omega = 0)}\left[\frac{1 - \kappa + b_i}{\kappa - b_i}\right]
\end{aligned}
$$

Replacing the vale of b_i for the unbiased and biased receivers yield the cutoffs π^* and π^{**}. For any $\frac{\pi(|0)}{\pi(s|1)} \leq \pi^*$, both types of receivers take the high action. When $\frac{\pi(|0)}{\pi(s|1)} \in (\pi^*, \pi^{**}]$, only the biased receiver takes the high action. For $\frac{\pi(|0)}{\pi(s|1)} \geq \pi^{**}$, no one takes the high action. This completes the corollary. □

Proof of Lemma 9. The objective is to construct the sender's payoff and optimal information structure following a choice of target $t \in N$. We can use the standard concavification approach in this case. Note that Proposition 4 tells us that following a seed t, all receivers in G_t receive the signal. Consequently, the receivers do not require any additional incentive to spread information. All that the sender cares is the proportion of different types of receivers in G_t.

Note that each signal from (S_t, π) induces a posterior distribution over the set of states Ω. Let p be the posterior probability that $\omega = 1$. Using Bayes' rule and Corollary 2 we can construct the sender's payoff

$$\tilde{v}(p) = \begin{cases} 0 & \text{if } p \in [0, \kappa - b) \\ v(G_t \cap B) & \text{if } p \in [\kappa - b, \kappa) \\ v(G_t) & \text{if } p \in [\kappa, 1] \end{cases}$$

Following Kamenica and Gentzkow (2011), the sender's problem can be restated in terms of choice of posteriors $p \in [0,1]$ subject to Bayes' plausibility:

$$\max_{\nu \in \Delta([0,1])} \mathbb{E}_\nu[\tilde{v}(p)]$$
$$\text{S.t. } \mathbb{E}_\nu[p] = \mu_0(\omega = 0)$$

From Corollary 1 of Kamenica and Gentzkow (2011), the solution to the above optimization problem can be constructed using the concave closure of $\tilde{v}$ (denoted by $F(p)$). This value depends on the slope of the concave closure at $p = \mu_0(\omega = 1)$. By Assumption 3, we know that $\mu_0(\omega = 1) < \kappa - b$. Hence the slope of the concave closure at $\mu_0(\omega = 1)$ is given by $\beta = \left\{ \frac{v(G_t)}{\kappa}, \frac{v(G_t \cap B)}{\kappa - b} \right\}$

The sender's payoff is then given by $F(\mu_0(\omega = 1)) = \mu_0(\omega = 1)\beta$. This is provided in the figure below. □

Proof of Theorem 3. The proof is analogous to Theorem 1. Under verifiable communication, all information is revealed in equilibrium. Hence all receiver in G_t receive the information that is observed by the seed. Note that no receiver takes the "high" action upon the realisation of a l signal. Let $A_t \subset G_t$ be the set of receivers who take the high action upon receiving the h signal. Hence the influence of receiver t is given by

$$C_t(G,(S_t,\pi)) = Pr[h]|A_t|$$

The key to the set equivalence is to construct the one-one mapping between the measure of influence and the sender's payoff. Let A_t be the set of receivers who choose the high action when the sender chooses (S_t^*,π^*). This is exactly what $V(h)$ defined in Lemma **??** measures. We have the following:

$$\begin{aligned} C_t(G,(S_t^*,\pi^*)) &= Pr(h)|A_t| \\ &= [\mu_0(\omega=1)+\mu_0(\omega=1)\pi^*(h|0)]V(h) \\ &= F_{(S_t^*,\pi^*)}(\mu_0) \end{aligned}$$

Take any $C_t(G,(S_t^*,\pi^*)) \in X^*$. By definition we have

$$\begin{aligned} & C_t(G,(S_t^*,\pi^*)) \geq C_j(G,(S_j^*,\pi^*)) \;\; \forall\, j \in N \\ & \leftrightarrow F_{(S_t^*,\pi^*)}(\mu_0) \geq F_{(S_t^*,\pi^*)}(\mu_0) \;\; \forall\, j \in N \end{aligned}$$

Hence $(S_t^*,\pi^*) \in \chi$, and $\psi^*\Bigg((S_t^*,\pi^*)\Bigg) > 0$. The converse follows similarly. As a result,

$$\psi^*\Bigg((S_t^*,\pi^*)\Bigg) > 0 \leftrightarrow C_t(G,(S_t^*,\pi^*)) \in X^*$$

This concludes the proof. □